On the constants of the Bohnenblust–Hille and Hardy–Littlewood inequalities

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The very beginning...

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Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

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Littlewood's 4/3 inequality (1930):

For all bilinear forms $T : \ell_\infty^n \times \ell_\infty^n \to \mathbb{K}$ and every positive integer n,

$$\left(\sum_{i,j=1}^{n} |T(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2}\, \|T\| \, .$$

Moreover, the power 4/3 is optimal.

One year later...

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H. F. Bohnenblust and E. Hille (Annals, 1931):

There exists a sequence of positive scalars $\left(B_{\mathbb{K},m}^{\mathrm{mult}}\right)_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \left|T(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\mathrm{mult}} \|T\|$$

for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and every positive integer n. The exponent $\frac{2m}{m+1}$ is optimal.

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Very complicated recursive formula... But in any case:

$$B_{\mathbb{C},m}^{\mathrm{mult}} \leq (m-1)^{0.31}$$

Best known estimates

The best known (upper) formulas for the case of real and complex scalars, up to now, are:

F. Bayart, D. Pellegrino, J. Seoane (Advances in Mathematics, 2014):

$$B_{\mathbb{C},m}^{\mathrm{mult}} \leq \prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}},$$

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$$B_{\mathbb{R},m}^{\mathrm{mult}} \leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}} \quad \text{for} \quad 2 \leq m \leq 13,$$

$$B_{\mathbb{R},m}^{\mathrm{mult}} \leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}} \quad \text{for} \quad m \geq 14.$$

Complex multilinear BH: estimates for the constants

	Optimal	2014	2013	1995	1978	1931
$B_{\mathbb{C},3}^{\mathrm{mult}}$	?	1.2184	1.24	1.27	2	4.17
$B_{\mathbb{C},4}^{\mathrm{mult}}$	?	1.2889	1.32	1.44	2.83	6.73
$B_{\mathbb{C},5}^{\mathrm{mult}}$	?	1.3474	1.42	1.62	4	10.51
$B_{\mathbb{C},6}^{\mathrm{mult}}$	?	1.3978	1.47	1.83	5.66	16.09
$B_{\mathbb{C},7}^{\mathrm{mult}}$	?	1.4422	1.53	2.06	8	24.33
$B_{\mathbb{C},8}^{\mathrm{mult}}$	?	1.4821	1.58	2.33	11.32	36.45
$B_{\mathbb{C},9}^{\mathrm{mult}}$	?	1.5183	1.63	2.63	16	54.24
$B_{\mathbb{C},10}^{\mathrm{mult}}$	?	1.5515	1.68	2.96	22.63	80.29
$B_{\mathbb{C},100}^{\mathrm{mult}}$	?	2.5118	4.55	$1.56 \cdot 10^5$	$7.9 \cdot 10^{14}$	$8.14 \cdot 10^{15}$

Classical Hardy–Littlewood inequalities

For $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$, let $\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

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Theorem (G. Hardy, J.E. Littlewood, T. Praciano-Pereira, 1934/1981)

For $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ there exist a constant $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}} \geq 1$ such that, for all positive integers n and all continuous m-linear forms $T : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}} \, \|T\| \, .$$

The exponent $\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}$ is optimal.

- Hardy–Littlewood (1934), Praciano-Pereira (1981):

$$C_{\mathbb{R},m,\mathbf{p}}^{\text{mult}} \leq \left(\sqrt{2}\right)^{m-1};$$

$$C_{\mathbb{C},m,\mathbf{p}}^{\text{mult}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1};$$

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- G.A.–Pellegrino–Silva (JFA, 2014):

$$C_{\mathbb{C},m,\mathbf{p}}^{\mathrm{mult}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{2m(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B_{\mathbb{C},m}^{\mathrm{mult}}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}.$$

Theorem (G.A., Pellegrino 2017)

Let $m \geq 3$ be a positive integer and $2m(m-1)^2 < p \leq \infty$. Then

$$C_{\mathbb{C},m,p}^{\mathrm{mult}} \leq \prod_{j=2}^{m} \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}.$$

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Generalized Bohnenblust–Hille inequality: if $(q_1, \ldots, q_m) \in [1,2]^m$ are so that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2},$$

then there is $B^{\mathrm{mult}}_{\mathbb{K},m,(q_1,\ldots,q_m)} \geq 1$ such that

$$\left(\sum_{j_1=1}^{n} \cdots \left(\sum_{j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{1}{q_1}} \leq B^{\mathrm{mult}}_{\mathbb{K},m,(q_1,\ldots,q_m)} \|T\|$$

for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$, and all positive integers n.

- Albuquerque–Bayart–Pellegrino–Seoane (2014): For $1 \leq q_1 \leq \cdots \leq q_m \leq 2$,

$$B^{\mathrm{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)} \leq \left(\prod_{j=1}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{2m\left(\frac{1}{q_m}-\frac{1}{2}\right)} \times \left(\prod_{k=1}^{m-1} \left(\Gamma\left(\frac{3k+1}{2k+2}\right)^{\left(\frac{-k-1}{2k}\right)(m-k)} \prod_{j=1}^{k} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{2k\left(\frac{1}{q_k}-\frac{1}{q_{k+1}}\right)} \right).$$

- Another particular cases: J.R. Campos, W. Cavalcante, V.V Fávaro, D. Núñez-Alarcón, D. Pellegrino, and D.M. Serrano-Rodríguez.

Theorem (G.A., Pellegrino 2017)

Let $m \geq 2$ be a positive integer and $q_1, \ldots, q_m \in [1, 2]$. If $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2}$ and $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then

$$B^{\mathrm{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)} \leq \prod_{j=2}^{m} \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}.$$

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... and new techniques must be invented so that one day we can get the definitive solution to these problems.

Final comments

Conjecture

The optimal constants in the Bohnenblust–Hille inequality are universally bounded, irrespectively of the value of m. In the real case, the best constants should be precisely

$$B_{\mathbb{R},m}^{\text{mult}} = 2^{1-\frac{1}{m}}.$$

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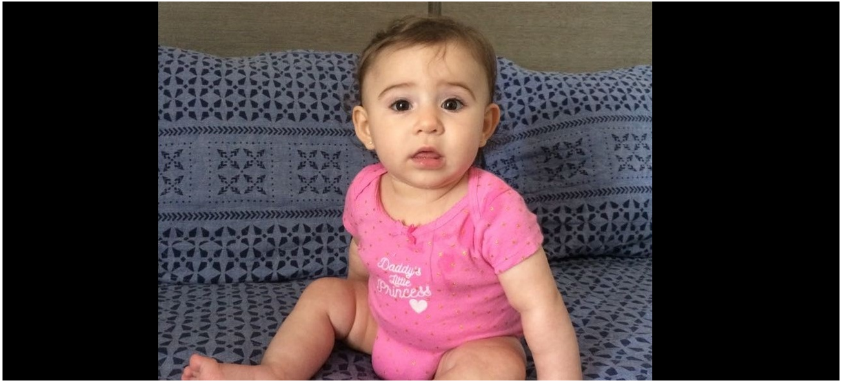
• T. Praciano–Pereira, On bounded multilinear forms on a class of ℓ_p spaces. J. Math. Anal. Appl. **81**(2) (1981), 561–568.

Thank you very much for your attention!



Theorem (Hölder's interpolative inequality for multiple exponents)

Let m, n, N be positive integers and $\mathbf{r}, \mathbf{q}(1), \ldots, \mathbf{q}(N) \in [1,\infty)^m$ and $\theta_1, \ldots, \theta_N \in [0,1]$ be such that $\theta_1 + \cdots + \theta_N = 1$ and

$$\frac{1}{r_j} = \frac{\theta_1}{q_j(1)} + \cdots + \frac{\theta_N}{q_j(N)}, \qquad \text{for all } j = 1, \ldots, m.$$

Then, for all scalar matrix $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i}}$ we have

$$\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \cdots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}} \leq \prod_{k=1}^{N} \left[\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}|^{q_m(k)} \right)^{\frac{q_{m-1}(k)}{q_m(k)}} \cdots \right)^{\frac{q_1(k)}{q_2(k)}} \right)^{\frac{1}{q_1(k)}} \right]^{\theta_k}.$$