On the Annihilator Ideal Graph

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Introduction

In [1], the zero-divisor graph of $R$ is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. This concept is due to Beck [2], who let all the elements of $R$ be vertices and was mainly interested in colorings.
Introduction


Introduction

- The zero-divisor graph of a ring $R$ has been studied extensively by many authors, for example see([3,4,5]).
Introduction


Introduction

In 2014, Badawi [6] introduced the annihilator graph of $R$. We recall from [6] that the annihilator graph of $R$ is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{Ann}(xy) \neq \text{Ann}(x) \cup \text{Ann}(y)$.
Introduction

Introduction

- It follows that each edge (path) of the classical zero-divisor of $R$ is an edge (path) of $AG(R)$. For Further investigations of $AG(R)$, see [7], [8]. The authors in [9] and [10] introduced the extended zero-divisor graph of $R$. 
We recall from [9] that the extended zero-divisor graph of $R$ is the undirected (simple) graph $EG(R)$ with the vertex set $Z(R)^*$, and two distinct vertices $x$ and $y$ are adjacent if and only if either $Rx \cap Ann(y) \neq \{0\}$ or $Ry \cap Ann(x) \neq \{0\}$. Hence it follows that the zero-divisor graph $\Gamma(R)$ is a subgraph of $EG(R)$.
Introduction


Introduction

A non-zero ideal $I$ of $R$ is called essential, denoted by $I \leq e R$, if $I$ has a non-zero intersection with any non-zero ideal of $R$. The ring $R$ is said to be reduced if it has no non-zero nilpotent element.
The socle of an R-module M, denoted by soc(M), is the sum of all simple submodules of M. If there are no simple submodules, this sum is defined to be zero. It is well known soc(M) is the intersection of all essential submodules. By dim(R) and depth(R), we mean the dimension and depth of R.
Introduction

- We write $\text{depth}(R) = 0$ if and only if every non-unit element of a ring is zero-divisor.

- We say $x$ is a regular element of $R$ if $x$ is non-unit and non zero-divisor.
Introduction

☐ The annihilator ideal graph of $R$, denoted by $\Gamma_{Ann}(R)$, is a graph

☐ whose vertices are all non-trivial ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if

☐ $I \cap \text{Ann}(J) \neq \{0\}$ or $J \cap \text{Ann}(I) \neq \{0\}$.
The diameter and girth of $\Gamma_{Ann}(R)$

- In this section, we study the diameter and the girth of the annihilator ideal graph of a ring.

- **Lemma 1.** Let $R$ be a ring. Then $\Gamma_{Ann}(R)$ is totally disconnected if and only if either $R$ is an integral domain or $R$ has only one non-zero proper ideal.
The diameter and girth of $\Gamma_{Ann}(R)$

- In the next theorem, it is proved that if $R$ is not an integral domain, then $\Gamma_{Ann}(R)$ is a connected graph of diameter at most 2.

- **Theorem 2.** Let $R$ be a ring. Then $\text{diam}(\Gamma_{Ann}(R)) \in \{0, 1, 2, \infty\}$. In particular, if $R$ is not an integral domain, then $\Gamma_{Ann}(R)$ is connected.
The diameter and girth of $\Gamma_{Ann}(R)$

The next corollary is an immediate consequence of Theorem 2.

Corollary 3. Suppose that $R$ is not an integral domain. Then $\Gamma_{Ann}(R)$ is a bipartite graph if and only if it is a complete bipartite graph.
The diameter and girth of $\Gamma_{Ann}(R)$

- To determine the girth of $\Gamma_{Ann}(R)$, the following lemma is needed.

- **Lemma 4.** Let $R$ be a non-reduced ring. Then there exists a vertex of $\Gamma_{Ann}(R)$ which is adjacent to every other vertex.

- **Theorem 5.** Let $R$ be a ring. Then $\text{gr}(\Gamma_{Ann}(R)) \in \{3, \infty\}$. 
The diameter and girth of $\Gamma_{Ann}(R)$

- In the next theorem, we study some relations between the diameters of
  - $\Gamma_{Ann}(R)$ and $\Gamma(R)$.
- **Theorem 6.** Let $R$ be a ring. Then the following statements hold:
  - (i) If $\text{diam}(\Gamma(R)) = 0$, then $\text{diam}(\Gamma_{Ann}(R)) = 0$.
  - (ii) If $\text{diam}(\Gamma(R)) = 1$, then $\text{diam}(\Gamma_{Ann}(R)) = 0$ or 1.
The diameter and girth of $\Gamma_{Ann}(R)$

- (iii) If $\text{diam}(\Gamma(R)) = 2$, then $\text{diam}(\Gamma_{Ann}(R)) = 1$ or 2.

- (iv) If $\text{diam}(\Gamma(R)) = 3$, then $\text{diam}(\Gamma_{Ann}(R)) = 1$ or 2.

- (v) If $\text{diam}(\Gamma_{Ann}(R)) = 0$ and $R$ is not an integral domain, then $\text{diam}(\Gamma(R)) = 0$ or 1.

- (vi) If $\text{diam}(\Gamma_{Ann}(R)) = 1$, then $\text{diam}(\Gamma(R)) = 1$ or 2 or 3.

- (vii) If $\text{diam}(\Gamma_{Ann}(R)) = 2$, then $\text{diam}(\Gamma(R)) = 2$ or 3.
Our main aim in this section is to study the annihilator ideal graphs with finite clique numbers. But first, it is shown that if $R$ is a reduced ring, then $\Gamma_{Ann}(R)$ is weakly perfect (Indeed, we show that $\Gamma_{Ann}(R)$ is a complete multipartite graph).

The following lemma will be used frequently in this paper.
A clique of $G$ is a complete subgraph of $G$ and the number of vertices in a largest clique of $G$, denoted by $w(G)$, is called the clique number of $G$. The chromatic number of $G$, denoted by $X(G)$, is the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. A graph $G$ is said to be weakly perfect if $w(G) = X(G)$.
Lemma 7. Let $R$ be a ring and $I; J \in \mathfrak{I}(R)$. Then the following statements hold.

1. If $I \prec J$ is not an edge of $\Gamma_{\text{Ann}}(R)$, then $\text{Ann}(I) = \text{Ann}(J)$. Moreover, if $R$

   is a reduced ring, then the converse is also true.

2. If $I \cap \text{Ann}(I) \neq (0)$, then $I$ is adjacent to every other vertex.

3. If $\text{Ann}(I) = (0)$ and $\text{Ann}(J) \neq (0)$, then $I \prec J$ is an edge of $\Gamma_{\text{Ann}}(R)$. 

8/1/2018
Annihilator Ideal Graph with Finite Clique Number

Let $R$ be a reduced ring. Using Lemma 2.1, we show that $\Gamma_{Ann}(R)$ is a complete multipartite graph.

Theorem 8. Let $R$ be a reduced ring. Then $w(\Gamma_{Ann}(R)) = \chi(\Gamma_{Ann}(R)) \in \{k; k+1\}$, where $k$ is the number of annihilator ideals of $R$. 
Annihilator Ideal Graph with Finite Clique Number

In two next results, we study rings whose annihilator ideal graphs have finite clique numbers.

Theorem 9. Let $R$ be a non-reduced ring, $w(\Gamma_{Ann}(R))<\infty$ and $I \leq e R$, for some ideal $I \subset Z(R)$. Then the following statements are equivalent.

1. $R$ is a Noetherian ring.
2. $R$ is an Artinian ring.
3. $\Gamma_{Ann}(R)$ is a complete graph.
Theorem 10. Let $R$ be a ring and suppose that $w(\Gamma_{Ann}(R)) < \infty$. Then the following statements are equivalent.

1. $Z(R) = \text{Nil}(R)$.
2. $R$ is an Artinian local ring.

Theorem 11. Let $R$ be a ring and $\text{depth}(R) \neq 0$. Then $w(\Gamma_{Ann}(R)) \neq 2$. 
A Main Subgraph of the Annihilator Graph of a Ring

- In this section, we study a subgraph of the annihilator ideal graph induced by ideals with non-zero annihilators. For instance, it is shown that $\Gamma_{Ann}(R)[I'(R)]$ is connected with diameter at most two and girth at most four (if it contains a cycle).
Recall that the annihilating-ideal graph of a ring $R$, denoted by $AG(R)$, is a graph with the vertex set $I'(R)$, and two distinct vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$. 
A Main Subgraph of the Annihilator Graph of a Ring

Theorem 12. Let $R$ be a ring. Then

- $\Gamma_{Ann}(R)[I'(R)]$ is connected and
- $\text{diam}(\Gamma_{Ann}(R)[I'(R)]) \leq 2$. Moreover, if $\Gamma_{Ann}(R)[I'(R)]$ contains a cycle, then
- $\text{girth}(\Gamma_{Ann}(R)[I'(R)]) \leq 4$. 
A Main Subgraph of the Annihilator Graph of a Ring

- The next theorem shows that \( \text{girth}(\Gamma_{Ann}(R)[I'(R)]) = 4 \) may occur.

- **Theorem 13.** Suppose that \( \Gamma_{Ann}(R)[I'(R)] \) contains a cycle. Then \( \text{girth}(\Gamma_{Ann}(R)[I'(R)]) = 4 \) if and only if \( R \) is reduced with \( |\text{Min}(R)| = 2 \).
A Main Subgraph of the Annihilator Graph of a Ring

- In order to characterize all rings $R$ whose $\Gamma_{Ann}(R)[I'(R)]$ is star, the following lemma is needed.
- **Lemma 14.** Let $R$ be a non-reduced ring. Suppose that $\Gamma_{Ann}(R)[I'(R)]$ is a star graph. Then the following statements hold.
  1. $R$ is indecomposable.
  2. $|I'(R)| = 2$. 
A Main Subgraph of the Annihilator Graph of a Ring

Theorem 15. Let $R$ be a ring. Then $\Gamma_{Ann}(R)[I'(R)]$ is a star graph if and only if one of the following statements holds.

1. $R \neq F \times D$, where $F$ is a field and $D$ is an integral domain.

2. $R$ is a local ring with exactly two non-trivial ideals.
Thanks for your attention