Calculus, heat flow and curvature-dimension bounds in metric measure spaces

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4. Curvature/dimension bounds
Goal and motivations

**Metric measure space (m.m.s.):** $(X, d, m)$, with $(X, d)$ complete and $m$ Borel measure, nonnegative, finite on bounded sets.

**Goal:** Can we extend to these abstract structures the classical notions of calculus (gradient, differential, Laplacian,.....), thus giving a precise meaning to ODEs, PDEs, and to differential geometric properties?

**Motivations:** Metric and Differential Geometry, Data Sets, Networks, Geometry of diffusion Operators,...
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A basic example of metric measure space

Weighted Riemannian manifold \((\mathbb{M}^n, g, m)\) with

- \(d = d_g\) Riemannian distance;
- \(m = e^{-V}\text{vol}_{\mathbb{M}^n}\).

The addition of the weight function \(e^{-V}\) makes the theory richer of applications and more “stable”. The Gauss space

\[
(\mathbb{R}^n, d_{eu}, \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} \mathcal{L}^n).
\]

(limit of \(n\)-dimensional linear projections of normalized surface measures on \(\sqrt{m}\mathbb{S}^n\) as \(m \to \infty\)) and the spaces

\[([0, \pi], d_{eu}, (\sin t)^{N-1} \mathcal{L}^1)\]

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Eulerian/Lagrangian duality
Many recent developments ultimately depend on the identification of concepts stated in Eulerian terms (dealing with gradients, Laplacians, Hessians,...) with those stated in Lagrangian terms (dealing with 1d curves). The relevance of these identifications is not sufficiently recognized when everything is smooth.

First example. One can compute the modulus of $\nabla f$, either by

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|},$$

or, in Lagrangian terms, by

$$|\nabla f|(x) = \sup \{ \partial_v f(x) : |v| = 1 \}$$

$$= \sup \{ d_x f(\gamma_t) : \gamma_t = x, \text{ Lip}(\gamma) \leq 1 \}.$$
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Second example: The ODE $\dot{\gamma}_t = b(t, \gamma_t)$, $\gamma_0 = x$ corresponds, in Eulerian terms, to the continuity equation

(PDE) $\partial_t u + \text{div}(bu) = 0 \quad (t, x) \in \mathbb{R} \times X$

because the flow map $X(t, x) := \gamma_t$ of the ODE provides the explicit formula

$$u(t, X(t, x)) = \frac{u(0, x)}{\det \nabla_x X(t, x)} \quad (t, x) \in \mathbb{R} \times X.$$

The study of the connections between (ODE) and (PDE), first when $b(t, \cdot)$ is not smooth (DiPerna-Lions), then in general metric measure structures (A-Trevisan), led to basic well-posedness and stability results.
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**Third example:** The incompressible Euler equations in fluid-mechanics are written as

\[ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p, \quad \text{div}_x \mathbf{v} = 0 \]

and, in Lagrangian terms, as

\[ \ddot{\gamma}_t = -\nabla p(t, \gamma_t). \]

Here, when \( \mathbf{v} \) and \( p \) are not smooth, the equivalence of the descriptions/notions of weak solution is much more subtle and much less understood (Brenier, Shnirelman, Sheffer, De Lellis-Székelyhidi, Isett, Buckmaster-Vicol,...).
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Weakly differentiable functions

Weak notions of derivative go back to the 1-d Calculus of Variations (Vitali, Tonelli, Lebesgue,...)

$$\min \left\{ \int_a^b L(t, \gamma_t, \dot{\gamma}_t) \, dt : \gamma_a = A, \gamma_b = B \right\}$$
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\min \left\{ \int_a^b L(t, \gamma_t, \dot{\gamma}_t) \, dt : \gamma_a = A, \gamma_b = B \right\}
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and to Dirichlet’s principle, for the solution of

\[-\Delta u = 0:

\min \left\{ \int_\Omega |\nabla u|^2 \, dx : u = g \text{ on } \partial\Omega \right\}.
\]

This led to the modern theory of Sobolev spaces, now a basic ingredient of PDE’s theory and Geometric Analysis.
Weakly differentiable functions

Many authors contributed to the development of this theory (Levi, Leray, Morrey, Evans, Schwartz, Sobolev, Fuglede,...) and now we recognize that, in the Euclidean space $\mathbb{R}^n$, the approaches

- distributional derivatives
- approximation by smooth functions
- good behaviour along almost all lines (curves)

are equivalent.
Weakly differentiable functions

**Distributional derivatives.** \( f \in W^{1,p}(\mathbb{R}^n) \) if \( f \in L^p(\mathbb{R}^n) \) and

\[
\int_{\mathbb{R}^n} f \nabla \phi \, dx = -\int_{\mathbb{R}^n} F \phi \, dx \quad \forall \phi \in C_c^\infty(\mathbb{R}^n)
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for some vector field \( F \in L^p(\mathbb{R}^n; \mathbb{R}^n) \) (the weak derivative).

**Approximation by smooth functions.** \( f \in H^{1,p}(\mathbb{R}^n) \) if \( f \in L^p(\mathbb{R}^n) \) and there exist \( f_i \in C^\infty(\mathbb{R}^n) \), \( F \in L^p(\mathbb{R}^n; \mathbb{R}^n) \) with

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\lim_{i \to \infty} \int_{\mathbb{R}^n} |f_i - f|^p + |\nabla f_i - F|^p \, dx = 0.
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**Good behaviour along $p$-almost all curves.** (Fuglede) $f \in BL^{1,p}(\mathbb{R}^n)$ if $f \in L^p(\mathbb{R}^n)$ and there exists $F \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ with

$$f(\gamma_1) - f(\gamma_0) = \int_{\gamma} F \quad \text{for } p\text{-almost every curve } \gamma : [0, 1] \to \mathbb{R}^n.$$

Beppo Levi first introduced this concept in 1901, restricting to straight lines in the coordinate directions. Here the precise meaning of “$p$-almost every curve” appeals to the notion of $p$-Modulus, a capacitary notion (Ahlfors-Beurling).
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Understanding properly what we mean by “smooth function”, “vector field”, “gradient” (Weaver, Cheeger, Hajlasz, Koskela-MacManus, Shanmugalingam,...) in the m.m.s. setting, leads to the following result:

**Theorem.**
In any m.m.s. \((X, d, m)\), the three definitions, properly adapted, are equivalent and define the same “gradient”.

The proof in full generality of this statement (A-Gigli-Savaré, Di Marino) requires tools (Hopf-Lax semigroup, superposition principle) providing “bridges” between these viewpoints, as well as key ideas from Optimal Transport.
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Background on Optimal Transport

The theory of Optimal Transport, pioneered by Monge (1781) and Kantorovich (1939) has been mostly “confined” for many years in the areas of Probability, Statistics, Linear Programming, Optimization.

In the last two decades, many more connections and applications emerged, in PDE’s, Functional and Geometric inequalities, and (even more recently) Metric Geometry, thanks to:

• new “nonlinear” interpolation between probability measures (Mc Cann)
• new “geometry” of the space of probability measures (Brenier, Otto).
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Monge’s formulation

Let $\mu, \nu \in \mathcal{P}(X)$ and $c : X \times X \rightarrow [0, \infty)$ a Borel cost function.
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Minimize $\int_X c(x, T(x)) \, d\mu(x)$ among all transport maps $T : X \to X$ pushing $\mu$ to $\nu$ (in short $T_\# \mu = \nu$), i.e.

$$\mu(T^{-1}(E)) = \nu(E) \quad \forall E \in \mathcal{B}(X).$$
Kantorovich’s formulation

Minimize \( \int_{X^2} c(x, y) \, d\Sigma(x, y) \) among all couplings \( \Sigma \in \mathcal{P}(X \times X) \) of \( \mu \) and \( \nu \), i.e.

\[ \Sigma(A \times X) = \mu(A), \quad \Sigma(X \times B) = \nu(B). \]

So, \( \Sigma(A \times B) \) is the amount of mass at \( A \), shipped to \( B \).

In \( \mathcal{P}_p(X) := \{ \mu : \int_X d^p(\cdot, \bar{x}) \, d\mu < \infty \} \), \( 1 \leq p < \infty \), the choice \( c = d^p \) induces the “nonlinear” \( L^p \) distances (Kantorovich-Rubinstein, Wasserstein)

\[ W_p(\mu, \nu) := \left( \min_{\Sigma} \int_{X^2} d^p(x, y) \, d\Sigma(x, y) \right)^{1/p} \]

which metrize weak convergence in \( \mathcal{P}(X) \), plus convergence of \( p \)-th moments.
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McCann’s displacement interpolation

In $\mathbb{R}^n$, if $T$ is an optimal transport map from $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ to $\nu \in \mathcal{P}_2(\mathbb{R}^n)$, then with $T_t = (1 - t)\text{Id} + tT$, one has the displacement interpolation

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\mu_t := (T_t)_#\mu \in \text{Geo}(\mathcal{P}_2(\mathbb{R}^n)) \quad \text{and} \quad T_t \text{ is optimal from } \mu \text{ to } \mu_t.
$$

With this new interpolation strategy in $\mathcal{P}_2(\mathbb{R}^n)$, McCann proved that Rényi’s entropy

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R_n(\mu) := -\int_{\mathbb{R}^n} \varphi^{1-1/n} \, dx
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is displacement convex and used this fact to establish a finer version of the Brunn-Minkowski inequality

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(L^n(A + B))^{1/n} \geq (L^n(A))^{1/n} + (L^n(B))^{1/n}.
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Dynamic formulation

More generally, we can focus on geodesic spaces \((X, d)\), with

\[
\text{Geo}(X) := \{ \text{constant speed geodesics } \gamma : [0, 1] \to X \}.
\]

The problem with \(c = d^2\) can now be written in terms of geodesic plans \(\eta\), namely probability measures on \(\text{Geo}(X)\):

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\text{Minimize } \int_{\text{Geo}(X)} \text{length}^2(\gamma) \ d\eta(\gamma) \quad \text{among all } \eta \in \mathcal{P}(\text{Geo}(X)) \text{ from } \mu \text{ to } \nu.
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The optimal transport problem canonically induces probability measures on \(\text{Geo}(X)\). Conversely, geodesics in \(\mathcal{P}_2(X)\) are all induced by probability measures in \(\text{Geo}(X)\)!
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Otto calculus

\[ \mathcal{P}_2(\mathbb{R}^n) \]
Otto calculus

Having in mind the continuity equation $\partial_t \mu + \text{div} (\mathbf{v} \mu) = 0$, Otto linked infinitesimal variations $s \in T_\mu \mathcal{P}_2(\mathbb{R}^n)$ to gradient velocities $\mathbf{v} = \nabla \phi$ and defined a (formal) Riemannian metric

$$g_\mu(s, s') := \int_{\mathbb{R}^n} \langle \nabla \phi, \nabla \phi' \rangle d\mu - \text{div} (\mu \nabla \phi) = s, \quad -\text{div} (\mu \nabla \phi') = s'.$$

It turns out (Benamou-Brenier) that the induced Riemannian distance

$$d^2_g(\mu_0, \mu_1) := \min \left\{ \int_0^1 \int_{\mathbb{R}^n} |\mathbf{v}(t, \cdot)|^2 d\mu_t \, dt : \partial_t \mu + \text{div} (\mathbf{v} \mu) = 0 \right\}$$

is precisely $W^2_2(\mu, \nu)$!
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g_\mu(s, s') := \int_{\mathbb{R}^n} \langle \nabla \phi, \nabla \phi' \rangle d\mu - \text{div} (\mu \nabla \phi) = s, -\text{div} (\mu \nabla \phi') = s'.
\]

It turns out (Benamou-Brenier) that the induced Riemannian distance

\[
d_2^2(\mu_0, \mu_1) := \min \left\{ \int_0^1 \int_{\mathbb{R}^n} |v(t, \cdot)|^2 d\mu_t dt : \partial_t \mu + \text{div} (v \mu) = 0 \right\}
\]

is precisely \( W^2_2(\mu, \nu) \)!
Heat Flow

The heat equation

\[ \partial_t u = \Delta u, \quad u \geq 0, \quad u(0, \cdot) = u_0, \quad \int_{\mathbb{R}^n} u_0(x) \, dx = 1 \]

can be understood from many and equally important points of view.

The probabilistic one views \( u(t, \cdot) \mathcal{L}^n \) as the law of the standard Brownian motion \( X^{u_0}_t \) starting from \( u_0 \):

\[ \int_{\mathbb{R}^n} u(t, x) \phi(x) \, dx = \mathbb{E} \left( \phi(X^{u_0}_{2t}) \right) \quad \forall \phi \geq 0. \]

The functional-analytic one (which does not need the sign and normalization restrictions) is related to the theory of gradient flows and semigroups.
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Given a Riemannian manifold \((\mathbb{M}, g)\) and \(S : \mathbb{M} \rightarrow \mathbb{R}\), the metric \(g\) converts \(d_x S\) into a vector \(\nabla S(x)\)

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g(\nabla S(x), v) = d_x S(v) \quad \forall v \in T_x \mathbb{M}
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and gives meaning to the (downward) gradient flow equation \(\dot{\gamma}_t = -\nabla S(\gamma_t)\).

Then, the functional-analytic interpretation of the heat equation is that \(u(t, \cdot)\) is the gradient flow in the \(L^2\) metric of Dirichlet’s energy:

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\mathcal{D}(u) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.
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However, at the end of the ’90, Jordan-Kinderlehrer-Otto realized that we can obtain another “Lagrangian” representation of the heat flow as the gradient flow of the Entropy Functional

\[
S(\mu) := \begin{cases} 
\int_{\mathbb{R}^n} \varrho \log \varrho \, dx & \text{if } \mu = \varrho \mathcal{L}^n; \\
+\infty & \text{otherwise}
\end{cases}
\]

in \( \mathcal{P}_2(\mathbb{R}^n) \) with respect to the distance \( W_2 \! \). One of the ways to realize this is to use the (formal) Riemannian structure of \( \mathcal{P}_2(\mathbb{R}^n) \), which provides the formula

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This discovery led to many more interpretations of conservative PDE’s

\[
\partial_t \rho = \text{div} \left( \rho \nabla U'(\rho) + \rho \nabla V + \rho \nabla (W * \rho) \right)
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(including Fokker-Planck, porous medium, interacting particle systems, etc.) as gradient flows of suitable “entropies”, generating an enormous literature on stability, trends of convergence to equilibrium,…

A key property, which will play also a role in the abstract and geometric setting, is that now the roles of \(d\) and \(m\) are nicely decoupled, unlike what happens for \(D\):

- \(d\) enters only in the computation of the optimal transport distance \(W_2\),
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Back to metric measure spaces

In m.m.s. \((X, d, m)\) the JKO interpretation makes perfectly sense, now with the Relative Entropy Functional (Kullback-Leibler divergence)

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\text{Ent}_m(\mu) := \begin{cases} 
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(and using a “metric” notion of gradient flow, due to De Giorgi).

But, also the \(L^2\)-gradient flow interpretation makes sense, replacing Dirichlet’s energy \(D\) with the so-called Cheeger energy:

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\text{Ch}(u) := \inf \left\{ \liminf_{h \to \infty} \int_X |\nabla u_h|^2 \, dm : u_h \in \text{Lip}(X, d), \|u_h - u\|_2 \to 0 \right\}.
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By “differentiating” Ch one obtains also a laplacian $\Delta = \Delta_{d,m}$, consistent with Witten’s laplacian $\Delta_g u - g(\nabla V, \nabla u)$ of weighted Riemannian manifolds and Finsler’s laplacian (thus, possibly nonlinear), defining the heat flow as the solution to

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**Theorem. (A-Gigli-Savaré)**

Under a mild regularity assumption on $\text{Ent}_m$ (fulfilled in all $CD(K, \infty)$ m.m.s.), one has $\mu_t = u_t m$ for all $t \geq 0$. 
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Curvature/dimension bounds

Bounds on the Ricci tensor are at the heart of many functional/geometric inequalities, in the broad field of comparison geometry, i.e. comparing with the model $N$-dimensional spaces with constant curvature. In particular, we deal with the lower bound $\text{Ric}_m := \text{Ric}_g + \nabla^2 V \geq Kg$, for $m = e^{-V} \text{vol}$, and the upper bound $N$ on dimension.

Example 1. (Spectral gap and Poincaré inequality)

$$\int_X (f - \overline{f})^2 \, dm \leq \frac{N - 1}{NK} \int_X |\nabla f|^2 \, dm, \quad K > 0.$$  

Example 2. (Gaussian isoperimetric inequality)

$$\text{Area}(\partial A) \geq \sqrt{KI}(\text{m}(A)) \quad A \subset X \text{ closed}, \quad K > 0,$$

where $I$ is the isoperimetric profile of the Gaussian space $\left(\mathbb{R}, d_{eu}, \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2} \, dx \right)$.  

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Example 3. (Levy-Gromov inequality) If $K > 0$,

$$\frac{m^+(E)}{m(X)} \geq \frac{|\partial B|}{|S|},$$

where $m^+(E) := \liminf_{r \downarrow 0} \frac{m(E_r) - m(E)}{r}$ is the Minkowski content of $E$ and $B$ is a spherical cap in $S = S^N_{\sqrt{K/(N-1)}}$ with $|B|/|S| = m(E)/m(X)$. 
Synthetic theories

The need of synthetic theories, going beyond the (weighted) Riemannian setting, emerges mainly from two independent directions: geometry of diffusion operators, limits of Riemannian manifolds.

**Diffusion operators.** Given a diffusion operator $L$ in $D(L) \subset L^2(X, m)$, one defines the carré du champ

$$\Gamma(u, v) := \frac{1}{2}(L(uv) - uLv - vLu) \quad (= g(\nabla u, \nabla v) \text{ if } L = \Delta_g)$$

and then the metric structure provided by the intrinsic distance $d_L$:

$$d_L(x, y) := \sup \{|f(x) - f(y)| : f \in D(L), \quad \Gamma(f, f) \leq 1\}.$$  

Witten Laplacian $Lf = \Delta_g f - g(\nabla V, \nabla f) \leadsto d_L = d_g$.

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Gromov-Hausdorff convergence for metric spaces (Gromov, ’81), and measured Gromov-Hausdorff convergence (Fukaya, ’87) are by now classical analytic tools in metric and Riemannian spaces.

Limits of Riemannian manifolds. By Gromov’s precompactness theorem, geometrically relevant m.m.s. can arise as (measured) Gromov-Hausdorff limits of Riemannian manifolds \( M^n \), for instance when

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**Example.** (collapsed limit)
Synthetic theories

In the ’90, Cheeger-Colding developed a remarkable and systematic program aimed at the description of the so-called Ricci limit spaces (with more recent contributions by Colding-Naber, Cheeger-Jiang-Naber), which includes:

- stratification of \( m \)-almost all of \( X \) in \( j \)-dimensional “regular” pieces \( R_j, 1 \leq j \leq n \);
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The synthetic theory aims at the description of a more general class of objects “from inside”, i.e. without referring to a smooth approximation (if any).

Compare with Alexandrov’s purely metric theory of upper and lower bounds on sectional curvature, based on triangle comparison.
Curvature-dimension: the Bakry-Émery theory

In the (weighted) Riemannian context, Ricci curvature appears in two basic formulas, an Eulerian and a Lagrangian one.

Bochner’s identity.

\[ \frac{1}{2} \Delta_g |\nabla f|^2 - \langle \nabla f, \nabla \Delta_g f \rangle = |\text{Hess}(f)|^2 + \text{Ric}_g(\nabla f, \nabla f). \]

In the context of diffusion operators \(L\), this leads to the Bakry-Émery BE\((K, N)\) condition

\[ \frac{1}{2} L\Gamma(f, f) - \Gamma(f, Lf) \geq \frac{(Lf)^2}{N} + K\Gamma(f, f). \]

When \(L\) is Witten’s laplacian, this is equivalent to a lower bound on the \((V, N)\)-modified Ricci tensor

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Curvature-dimension: the Bakry-Émery theory

The “calculus” encoded in $\text{BE}(K, N)$ works even in structures very far from Riemannian manifolds, as (infinite-dimensional) Gaussian spaces.

This leads to synthetic proofs of many functional/geometric inequalities, very often with sharp constants (Bakry, Ledoux, Gentil, Bobkov, Baudoin, Garofalo,...).

In the case $N = \infty$, gradient contractivity and Logarithmic Sobolev Inequality

$$|\nabla P_t f|^2 \leq e^{-2Kt} P_t |\nabla f|^2, \quad \int_X f^2 \log f^2 \, dm \leq \frac{2}{K} \int_X |\nabla f|^2 \, dm$$

as well as Transport inequalities, Gaussian isoperimetric inequalities, etc.

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Ricci curvature and displacement convexity

In the Riemannian setting, the link between Ricci curvature and displacement convexity in $\mathcal{P}_2(X)$ goes back to the work of Cordero-McCann-Schmuckenschläger, Otto-Villani, Sturm-Von Renesse.

It relies on a suitable $(K, N)$-concavity inequality satisfied by the Jacobian function $\mathcal{J}(\cdot, x)$ along geodesics

$$\mathcal{J}(s, x) := \det [\nabla_x T_s(x)] \quad \text{with} \quad T_s(x) := \exp(s \nabla \phi(x)).$$

For instance, in the limit as $N \to \infty$, the inequality

$$\log \mathcal{J}(s, x) \geq s \log \mathcal{J}(1, x) + (1 - s) \log \mathcal{J}(0, x) + K \frac{s(1 - s)}{2} |\nabla \phi|^2(x).$$
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Curvature-dimension: the Lott-Villani and Sturm theory

In the nonsmooth world, the basic idea (Lott-Villani, Sturm) is to average these inequalities along the geodesics selected by the Optimal Transport problem with cost=distance².

**CD(Κ, N) condition.** It is defined through a (Κ, N)-convexity inequality, along constant speed geodesics of (P₂(X), W₂) involving Rényi’s entropies

\[ U_{N,m}(\mu) = - \int_X \varrho^{1-1/N} \, dm \quad \text{if} \quad \mu = \varrho m \in P_2(X). \]

In the simpler case Ν = ∞ one uses their (rescaled) limit, the relative entropy w.r.t. m:

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Advantages of the CD theory

- Stability w.r.t. measured Gromov-Hausdorff convergence and its many variants, ensured by
  
  (a) the decoupling of the roles of $d$ and $m$;
  
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- While the Bakry-Emery theory is more “Riemannian”, the CD theory is able to cover more general classes of spaces, as Finsler manifolds.

- As for the Bakry-Emery theory, it provides synthetic proofs of many functional inequalities and comparison results: Bishop-Gromov, Bonnet-Myers, spectral gap, Laplacian comparison, etc. (Lott-Villani, Sturm, Gigli, Cavalletti-Mondino,...)
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Two striking results of the CD theory

Representing the heat flow as $W_2$-gradient flow of $\text{Ent}_m$ gives:

**Theorem. (Gigli-Mondino-Savaré)**

Assume that $(X^n, d^n, m^n)$ are CD$(K, \infty)$ and m-GH converge to $(X, d, m)$. Then the Cheeger energies $Ch^n$, the heat flows $P^n_t$, the Laplacians $\Delta^n$ all converge to $Ch$, $P_t$, $\Delta$ respectively.

The second result follows by an adaptation of the localization technique (Klartag, Gromov-Milman, Kannan-Lovász-Simonovitz) to m.m.s.:

**Theorem. (Cavalletti-Mondino)**

The Levy-Gromov inequality holds in CD$(K, N)$ non-branching spaces (even, when $K < 0$, with a very general class of model spaces singled out by E.Milman).
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Adding the “Riemannian” assumption

We say that \((X, d, m)\) is infinitesimally Hilbertian if Cheeger’s Dirichlet energy is quadratic, namely

\[
Ch(f + g) + Ch(f - g) = 2Ch(f) + 2Ch(g) \quad \forall f, g.
\]

This class includes Riemannian manifolds and Alexandrov spaces, it rules out Finsler spaces.

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RCD(K, N) = CD(K, N) + \text{Infinitesimally Hilbertian} \quad (\text{Gigli, 2012})
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**Theorem. (A.-Gigli-Savaré)**

The \(RCD(K, N)\) condition is stable under \(m\)-GH convergence. Moreover, \(RCD(K, \infty)\) is encoded by a differential inequality satisfied by the heat flow \(P_t\):

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Equivalence of BE and RCD

Since the BE and CD theories both provide synthetic upper bounds on dimension and lower bounds on Ricci curvature, it is natural to guess that they should be more closely related, at least under the Riemannian assumption.

Thm. (A.-Gigli-Savaré, Erbar-Kuwada-Sturm, A.-Mondino-Savaré)

In the class of infinitesimally Hilbertian m.m.s. the BE($K$, $N$) and the CD($K$, $N$) theories are (essentially) equivalent.

The advantages of this identification are obvious: one can use the strength of both theories, playing at the Eulerian and Lagrangian levels at the same time.
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• Bridge the gap between Ricci limit and $\text{RCD}(K, N)$ spaces;
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