Optimization of Lyapunov Exponents

Jairo Bochi (PUC–Chile)

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Part 1

Commutative ergodic optimization: Birkhoff averages

References: Surveys by O. Jenkinson.

- *Ergodic Optimization in Dynamical Systems*, Ergodic Theory Dynam. Systems (2018; online)

Apology / Disclaimer: I won’t discuss relations with Lagrangian Mechanics, nor Thermodynamical Formalism.
General setting for the whole talk

- $X = \text{compact metric space}$
- $T: X \to X$ continuous map
- $\mathcal{M}_T := \text{set of } T\text{-invariant Borel probability measures}$
  (compact convex)
- $\mathcal{M}_T^{\text{erg}} := \text{subset of ergodic measures } = \text{ext}(\mathcal{M}_T)$. 
Ergodic optimization of Birkhoff averages

Given a continuous function $f : X \rightarrow \mathbb{R}$ ("potential"),

$$\left\{ \int f \, d\mu ; \mu \in \mathcal{M}_T \right\} =: [\alpha(f), \beta(f)]$$

level sets of $\mu \mapsto \int f \, d\mu$
Ergodic optimization of Birkhoff averages

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\[
\left\{ \int f \, d\mu \;; \mu \in \mathcal{M}_T \right\} =: [\alpha(f), \beta(f)]
\]

\[\beta(f)\]

$\mu_{\text{max}}$

level sets of $\mu \mapsto \int f \, d\mu$

$\alpha(f)$

$\mu_{\text{min}}$

$\mu \in \mathcal{M}_T$ s.t. $\int f \, d\mu = \beta(f)$ is called a \textbf{maximizing measure}.

Note: \textbf{Ergodic} maximizing measures always exist. In particular, uniqueness $\Rightarrow$ ergodicity.
Expressing $\beta(f)$ in terms of Birkhoff averages

Birkhoff sum $f^{(n)} := f + f \circ T + \cdots + f \circ T^{n-1}$

$$\beta(f) = \sup_{x \in X} \limsup_{n \to \infty} \frac{f^{(n)}(x)}{n}$$

$$= \lim_{n \to \infty} \sup_{x \in X} \frac{f^{(n)}(x)}{n}$$
Ergodic optimization of Birkhoff averages

Meta-Problem

To understand maximizing measures.
Maximizing measures: Generic uniqueness

Theorem (Conze–Guivarch, Jenkinson, . . .)

Let $\mathcal{F}$ be any “reasonable” space $\mathcal{F}$ of continuous functions. For generic $f$ in $\mathcal{F}$, the maximizing measure is unique.
Theorem (Conze–Guivarch, Jenkinson, . . .)

Let $F$ be any “reasonable” space $F$ of continuous functions.
For generic $f$ in $F$, the maximizing measure is unique.

“Reasonable” space: a topological vector space $F$ continuously and densely embedded in $C^0(X)$.

Generic property: a property that holds on a dense $G_δ$ subset (of a Baire space).
The inverse problem

Theorem (Jenkinson)

Given $\mu \in \mathcal{M}_T^{\text{erg}}$, there exists $f \in C^0(X)$ such that $\mu$ is the unique maximizing measure for $f$. 
The inverse problem

Theorem (Jenkinson)

\[ \text{Given } \mu \in M^\text{erg}_T, \text{ there exists } f \in C^0(X) \text{ such that } \mu \text{ is the unique maximizing measure for } f. \]

If \( \mu \) has finite support then \( f \) can be taken \( C^\infty \).

For a general \( \mu \), how regular \( f \) can be taken? Not much...
Maximizing sets

Assume the following **nice setting**:

- $T : X \to X$ is "**hyperbolic**" (e.g. uniformly expanding, Anosov);
- $f : X \to \mathbb{R}$ is "**regular**" (at least Hölder).
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**Theorem (Subordination principle)**

*In this nice setting, there is a maximizing set*: a \( T \)-invariant compact set \( K \subseteq X \) such that

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\mu \text{ is maximizing} \iff \text{supp}\ \mu \subseteq K
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- It is **false** if $f$ is only $C^0$ (by the previous theorem)
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- It is **false** if $f$ is only $C^0$ (by the previous theorem)
- It is a corollary of the **Mañé Lemma** (or **Revelation Lemma** or **Nonpositive Livsic Lemma**).

Suppose $T : X \rightarrow X$ is chaotic.

Then for typical regular functions $f : X \rightarrow \mathbb{R}$, the maximizing measure has low complexity.
Expected panorama for the nice setting


Suppose \( T : X \to X \) is chaotic (unif. expanding / unif. hyperbolic / \ldots). Then for typical regular functions \( f : X \to \mathbb{R} \), the maximizing measure has low complexity.

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Suppose $T : X \to X$ is chaotic (unif. expanding / unif. hyperbolic / . . .). Then for typical (topological sense / probabilistic sense) regular (Hölder / . . . / analytic) functions $f : X \to \mathbb{R}$, the maximizing measure has low complexity.

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Known results in this direction


Suppose the dynamics $T : X \to X$ is chaotic. Then for typical regular functions $f : X \to \mathbb{R}$, the maximizing measure has low complexity.

Many results (including Yuan, Hunt’99; Contreras, Lopes, Thieullen’01; Bousch’01; Morris’08; Quas, Siefken’12); the best one is:
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Theorem (Contreras’16)

$T$ unif. expanding $\Rightarrow$ for generic Lipschitz $f$’s (actually all $f$’s in an open and dense subset), the maximizing measure is supported on a periodic orbit.
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**Theorem (Contreras’16)**

$T$ unif. expanding $\Rightarrow$ for generic Lipschitz $f$’s (actually all $f$’s in an open and dense subset), the maximizing measure is supported on a periodic orbit.

Only result with a probabilistic notion of typicality (**prevalence**): B., Zhang’16.
A beautiful example

Conze, Guivarch’93; Hunt–Ott’96; Jenkinson’96; Bousch’00

\[ T(x) = 2x \mod 2\pi \text{ on the circle } X := \mathbb{R}/2\pi\mathbb{Z} \]

\( f = \text{trigonometric polynomial of deg. 1} \)

WLOG, \( f(x) = f_\theta(x) = \cos(x - \theta) \)
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**Theorem (Bousch’00)**

For every \( \theta \in [0, 2\pi] \), the function \( f_\theta \) has a unique maximizing measure \( \mu_\theta \), and it has zero entropy (actually, Sturmian).
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**Theorem (Bousch’00)**

*For every \( \theta \in [0, 2\pi] \), the function \( f_\theta \) has a unique maximizing measure \( \mu_\theta \), and it has zero entropy (actually, Sturmian).

*Furthermore, for Lebesgue-a.e. \( \theta \) (actually, all \( \theta \) outside a set of Hausdorff dim. 0), \( \mu_\theta \) is supported on a periodic orbit.*
Part 2
Non-commutative ergodic optimization:
Top Lyapunov exponent
Replace the scalar function $f$ by a (continuous) matrix-valued function:

$$F : X \rightarrow \text{Mat}(d \times d, \mathbb{R}) \text{ or } \text{GL}(d, \mathbb{R}) \quad \text{ ("cocycle").}$$

The Birkhoff sums are replaced by products:

$$F^{(n)}(x) := F(T^{n-1}x) \cdots F(Tx)F(x).$$

**Top Lyapunov exponent:**

$$\lambda_1(F, x) := \lim_{n \to \infty} \frac{1}{n} \log \|F^{(n)}(x)\| \quad \text{(if it exists)}$$

For any $\mu \in \mathcal{M}_T$, the limit exists for $\mu$-a.e. $x \in X$.

$$\lambda_1(F, \mu) := \int \lambda_1(F, x) \, d\mu(x)$$
Optimization of the top Lyapunov exponent

Quantities of interest:

\[ \alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) \]
\[ \beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) \]
Optimization of the top Lyapunov exponent

Quantities of interest:

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For “step cocycles”:

- \( e^{\beta(F)} \) is called **joint spectral radius** (Rota, Strang’60; Daubechies, Lagarias’92, . . .)
- \( e^{\alpha(F)} \) is called **joint spectral subradius** (Gurvits’95).

Another characterization:

\[ \beta(F) = \lim_{n \to \infty} \sup_{x \in X} \frac{1}{n} \log \| F^{(n)}(x) \| . \]
**Basic difficulty:**

\[ \mu \in \mathcal{M}_T \mapsto \lambda_1(F, \mu) \] is **not continuous**, in general.

It is **upper semi-continuous**, at least.
Basic difficulty:
$\mu \in \mathcal{M}_T \mapsto \lambda_1(F, \mu)$ is not continuous, in general.
It is upper semi-continuous, at least.

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) \quad \smile \text{ not necessarily attained}$$

$$\beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) \quad \heartsuit \text{ always attained}$$
Example without $\lambda_1$-minimizing measure

Step cocycle $T: \{0, 1\}^\mathbb{N} \leftrightarrow \text{shift}$, $F(x) = A_{x_0}$ where $A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Claim
\[ \alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) = -\log 2, \] but the inf is not attained.

Proof.
Example without $\lambda_1$-minimizing measure

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$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) = -\log 2$, but the inf is not attained.

Proof.

$A_1 A_0^n = \begin{pmatrix} 0 & -2^{-3n} \\ 2^n & 0 \end{pmatrix}$ has eigenvalues $\pm 2^{-2n} i$, so

$\mu_n := \delta(0^n 1)^\infty \Rightarrow \lambda_1(F, \mu_n) = -\frac{n}{n+1} \log 2 \downarrow -\log 2$.

So $\alpha(F) \leq -\log 2$. Discontinuity: $\lambda_1(F, \lim \mu_n) \neq \lim \lambda_1(F, \mu_n)$.

On the other hand...
Example without $\lambda_1$-minimizing measure

Step cocycle $T : \{0, 1\}^\mathbb{N} \leftrightarrow$ shift, $F(x) = A_{x_0}$ where $A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Claim

$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu) = - \log 2$, but the inf is not attained.

Proof.

\[
\lambda_1(F, \mu) \overset{(1)}{=} \frac{\lambda_1(F, \mu) + \lambda_2(F, \mu)}{2} = \int \frac{1}{2} \log |\det F(x)| \, d\mu(x) \overset{(2)}{=} - \log 2.
\]

So $\alpha(F) \geq - \log 2$ and therefore $\alpha(F) = - \log 2$. Moreover, (2) becomes "=" iff $\mu = \delta_{0^\infty}$, but then (1) is ">". So no $\mu$ attains $\lambda_1(F, \mu) = - \log 2$. 
Expected panorama for $\lambda_1$-maximization

Meta-Conjecture

Suppose $T$ is chaotic (unif. expanding / unif. hyperbolic / ...). Then for typical (topological sense / probabilistic sense) regular (Hölder / .../ analytic) cocycles $F$, the $\lambda_1$-maximizing measure has low complexity (zero topological entropy / .../ supported on a periodic orbit).

A result that fits this philosophy: B., Rams’16.
Some initial results

Similarly to the commutative **subordination principle**:

**Theorem (B., Garibaldi)**

*Suppose $T$ is a hyperbolic homeomorphism, and that $F$ is a strongly fiber-bunched cocycle. Then there exists a maximizing set: a $T$-invariant compact set $K \subseteq X$ such that*

\[ \mu \text{ is } \lambda_1\text{-maximizing} \iff \text{supp } \mu \subseteq K \]
Some initial results

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\]

This is actually a corollary of a **version of Mañé Lemma for cocycles** (existence of extremal norms), which has other applications.

Related work: Morris’10, Morris’13.
Part 3

Non-commutative ergodic optimization: Full Lyapunov spectra

**Extra information:** Proceedings paper (ArXiv 1712.01612)
The other Lyapunov exponents

\[ T : X \rightarrow X, \; F : X \rightarrow \text{GL}(d, \mathbb{R}) \text{ as before.} \]

For each \( i \in \{1, 2, \ldots, d\} \), and \( x \in X \), let

\[ \lambda_i(F, x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \log s_i(F^n(x)) \quad \text{(if it exists)} \]

where \( s_i(\cdot) := i\text{-th biggest singular value.} \)

For any \( \mu \in \mathcal{M}_T \), these limits exist for \( \mu\text{-a.e. } x \in X \).

If \( \mu \) is \textbf{ergodic}, then \( \lambda_i(F, \cdot) \) is \( \mu\text{-a.e. equal to some constant } \lambda_i(F, \mu) \).
Given $(T, F)$, the **Lyapunov vector** of $\mu \in \mathcal{M}_T^{\text{erg}}$ is:

$$\tilde{\lambda}(F, \mu) := (\lambda_1(F, \mu), \ldots, \lambda_d(F, \mu))$$

The **Lyapunov spectrum** of $(T, F)$ is:

$$L^+(F) := \{\tilde{\lambda}(F, \mu) ; \mu \in \mathcal{M}_T^{\text{erg}}\},$$

which is a subset of the **positive chamber**:

$$a^+ := \{ (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d ; \xi_1 \geq \cdots \geq \xi_d \}.$$
Lyapunov spectrum of a cocycle

\[ L^+(F) := \{ \lambda(F, \mu) ; \mu \in \mathcal{M}^\text{erg}_T \} \]

\[ \subset a^+ := \{ (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d ; \xi_1 \geq \cdots \geq \xi_d \} . \]
If $F$ takes values in $\text{SL}(3, \mathbb{R})$ then the Lyapunov spectrum is also contained in the plane

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 ; \xi_1 + \xi_2 + \xi_3 = 0\}$$

Related: Sert’17 has a notion of “joint spectrum” (more general Lie groups); he proves large deviation results.
A nice result (for the “nice setting”)

**Theorem (Kalinin’11)**

Suppose $T: X \to X$ is hyperbolic, and $F: X \to \text{GL}(d, \mathbb{R})$ is a Hölder-continuous cocycle. Then the Lyapunov vectors of measures supported on **periodic orbits** are dense in the Lyapunov spectra $L^+(F)$. 

Meta-Conjecture (Typical spectra; part 1)

Suppose $T : X \to X$ is hyperbolic, and $F : X \to \text{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

1. The Lyapunov spectrum $L^+(F)$ is a convex set.
2. Its boundary is "fishy".
3. Every boundary point $\vec{s}$ outside the walls is attained as the Lyapunov vector of a unique ergodic measure $\vec{s}$; furthermore, $\vec{s}$ has low complexity (zero topological entropy).
4. Subordination property: these $\vec{s}$ have uniquely ergodic supports.
Suppose $T : X \to X$ is hyperbolic, and $F : X \to \text{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

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Expected picture of $L^+(F)$

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A particular but concrete example

“Step cocycle” $T : \{0, 1\}^\mathbb{N} \leftrightarrow \text{shift}, F(x) = A_{x_0}$ where $A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$. Then:

$L_{+}(F)$ is convex. Its boundary is composed of a piece of the wall $s_1 = s_2$ and a curve with a dense subset of corners – “fishy.” Every point in this curve is attained as the Lyapunov vector of a unique ergodic measure, which is Sturmian.

Corollary of works by Hare, Morris, Sidorov, Theys’11; Morris, Sidorov’13 (on counterexamples for the “finiteness conjecture”; see also Bousch, Mairesse’01).
A particular but concrete example

“Step cocycle” $T: \{0, 1\}^\mathbb{N} \leftrightarrow$ shift, $F(x) = A x_0$ where $A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$. Then:

- $L^+(F)$ is convex.

- Its boundary is composed of a piece of the wall $\xi_1 = \xi_2$ and a curve with a dense subset of corners – “fishy”.

- Every point in this curve is attained as the Lyapunov vector of a unique ergodic measure, which is Sturmian.

Corollary of works by Hare, Morris, Sidorov, Theys’11; Morris, Sidorov’13 (on counterexamples for the “finiteness conjecture”; see also Bousch, Mairesse’01).
The simplest case

If \( F(x) = \begin{pmatrix} e^{f_1(x)} & 0 \\ 0 & e^{f_2(x)} \end{pmatrix} \) where \( f_1 > f_2 \) then:

- The Lyapunov vector \( \mu \mapsto \vec{\lambda}(F, \mu) \) is continuous, since it equals \( \int \vec{f} \, d\mu \) where \( \vec{f} = (f_1, f_2) \).
The simplest case

If \( F(x) = \begin{pmatrix} e^{f_1(x)} & 0 \\ 0 & e^{f_2(x)} \end{pmatrix} \) where \( f_1 > f_2 \), then:

- The Lyapunov vector \( \mu \mapsto \tilde{\lambda}(F, \mu) \) is continuous, since it equals \( \int \tilde{f} \, d\mu \) where \( \tilde{f} = (f_1, f_2) \).
- The Lyapunov spectrum \( L^+(F) \) is a “rotation set”; in particular it is compact and convex, and its extremal points are attained by ergodic measures.
- \( L^+(F) \) is away from the wall \( \xi_1 = \xi_2 \).
The simplest case

If \( F(x) = \begin{pmatrix} e^{f_1(x)} & 0 \\ 0 & e^{f_2(x)} \end{pmatrix} \) where \( f_1 > f_2 \) then:

- The Lyapunov vector \( \mu \mapsto \lambda(F, \mu) \) is continuous, since it equals \( \int \tilde{\mathbf{f}} \, d\mu \) where \( \tilde{\mathbf{f}} = (f_1, f_2) \).
- The Lyapunov spectrum \( L^+(F) \) is a “rotation set”; in particular it is compact and convex, and its extremal points are attained by ergodic measures.
- \( L^+(F) \) is away from the wall \( \xi_1 = \xi_2 \).

**Commutativity regained:** Essentially the same happens if the cocycle admits a dominated splitting into one-dimensional bundles – which is an open property.
The **rotation set** of a continuous $\vec{f} : X \to \mathbb{R}^d$ is:

$$R(\vec{f}) := \{ \int \vec{f} \, d\mu ; \mu \in \mathcal{M}_T \}$$

compact and convex; an affine projection of $\mathcal{M}_T$ in $\mathbb{R}^d$. 
A step back: vectorial ergodic optimization

The **rotation set** of a continuous \( \tilde{f} : X \to \mathbb{R}^d \) is:

\[
R(\tilde{f}) := \left\{ \int \tilde{f} \, d\mu \; ; \; \mu \in \mathcal{M}_T \right\}
\]

compact and convex; an affine projection of \( \mathcal{M}_T \) in \( \mathbb{R}^d \).

**Example (The fish: Jenkinson’96, Bousch’00)**

\( T(x) = 2x \text{ mod } 2\pi, \; \tilde{f}(x) = (\cos x, \sin x). \)

- Fishy boundary: dense subset of corners.
- Each corner comes from a periodic orbit.
- Boundary points come from low-complexity measures (Sturmian).

**Note:** No Mañé Lemma for vectorial ergodic optimization (B., Delecroix) – see Proceedings paper.
Back to cocycles: Dominated splittings

Suppose the cocycle $F: X \rightarrow \text{GL}(d, \mathbb{R})$ admits an \textbf{invariant} splitting:

$$\mathbb{R}^d_x = V_x \oplus W_x \quad \begin{align*} F(x)(V_x) &= V_{Tx}, \quad F(x)(V_x) = W_{Tx}. \end{align*}$$

The splitting is \textbf{dominated} if $\exists c \in (0, 1)$ s.t. (changing the norm if necessary)

$$\|F(x)w\| < c\|F(x)v\| \quad \forall x, \quad \forall \text{unit vectors } v \in V_x, \quad w \in W_x.$$  

$(\Leftrightarrow \text{uniform exponential separation of } \text{singular values } s_i, s_{i+1} \text{ for the products } F^n(x): \text{B., Gourmelon'09})$
Every cocycle admits a **finest dominated splitting** \( \mathbb{R}^d = V_1 \oplus V_2 \oplus \cdots \oplus V_k \) (maybe **trivial** \( k = 1 \)).

If the splitting is **simple** \( k = d \) then we recover commutativity.

**Possible strategy to obtain convexity of** \( L^+(F) \): use subsystems with simple dominated splitting?
Domination vs. Lyapunov exponents

If a cocycle admits a dominated splitting with dominating bundle of dim. $i$ then the Lyapunov spectrum $L^+(F)$ is away from the wall $\xi_i = \xi_{i+1}$.

⚠️ The converse is false . . .
If a cocycle admits a dominated splitting with dominating bundle of dim. $i$ then the Lyapunov spectrum $L^+(F)$ is **away from the wall** $\xi_i = \xi_{i+1}$.

The converse is **false** ... but maybe true for typical cocycles? (known counterexamples are too delicate)
Meta-Conjecture (Typical Lyapunov spectra – continued)

Suppose $T : X \to X$ is hyperbolic, and $F : X \to \text{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

1. The Lyapunov spectrum $L^+ (F)$ is a convex set.
2. Its boundary is “fishy” (dense subset of corners away from walls).
3. Every boundary point $\xi$ outside the walls is attained as the Lyapunov vector of a unique ergodic measure $\mu_\xi$; furthermore, $h(\mu_\xi, T) = 0$.
4. Subordination property: these $\mu_\xi$ have uniquely ergodic supports.
5. $L^+ (F)$ touches the wall $\xi_i = \xi_{i+1}$ iff there is no dominated splitting with dominating bundle of dim. $i$. 
Extra convexity properties of $L^+(F)$?

Let’s add still another item:

**Meta-Conjecture (Typical Lyapunov spectra – continued)**

Suppose $T : X \to X$ is hyperbolic, and $F : X \to \text{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

...  

6. There exists a (larger) **convex set** $M^+(F) \subset \mathbb{R}^d$ (Morse set) such that $M^+(F) \cap \alpha^+ = L^+(F)$ and $M^+(F)$ is invariant by reflections across the walls it touches.

**Remark:** The terminology **Morse set** comes from Control Theory: Colonius, Kliemann’96, ’02 – chain transitivity on projective and flag bundles.
Extra convexity properties of $L^+(F)$?

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$(F$ in $\text{SL}(3, \mathbb{R})$; no dominations)
Rationale for (6) extra convexity

**Philosophy:** Lack of domination (of “index” \( i \)) should allow us to mix (make convex combinations) of Lyapunov exponents \( \lambda_i \) and \( \lambda_{i+1} \).

**Example (seen before)**

The step cocycle induced by matrices \( \begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix} \) and \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).
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The step cocycle induced by matrices $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{8} \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Some implementations of this “philosophy”: B.’01; B., Viana’05; B., Bonatti’12 (perturbative). Gorodetski, Ilyashenko, Kleptsyn, Nalsky’05; B., Bonatti, Díaz’14, ’16 (non-perturbative).

⚠️ On the other hand, if a conjecture by B., Fayad’06 is true then (6) is false for probabilistic-typical step cocycles in dim. 2. (But step cocycles don’t look very typical...)