Dimension of Self-Similar Sets and Measures: A Survey of Recent Results

Michael Hochman

Einstein Institute of Mathematics
The Hebrew University of Jerusalem

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Self-similar measures - motivation via random walks

- $\Phi = \{f_1, \ldots, f_k\}$ similarity maps of $\mathbb{R}^d$, no common fixed point.
- $p = (p_1, \ldots, p_k)$ a positive probability vector (a measure on Φ).
- Define a random walk on $\mathbb{R}^d$ by fixing a point $x_0 \in \mathbb{R}^d$ and setting

$$x_{n+1} = \xi_{n+1} x_n$$

where $\xi_1, \xi_2, \xi_3, \xi_4, \ldots$ are independent similarities with distribution p.

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f_i contract	$\Longrightarrow$	distribution$(x_n) \to \mu$ (no universal behavior).
f_i expand	$\Longrightarrow$	$x_n \to \infty$, Boundary theory (no universal behavior)

Here we focus mainly on the contracting case

The limit distribution μ is called a self-similar measure:

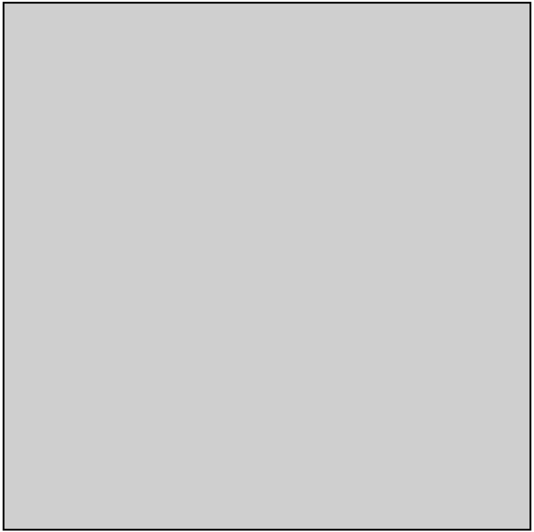
$$\mu = \sum_{i=1}^{k} p_i \cdot f_i\mu$$

($f_i\mu = \mu \circ f_i^{-1}$ is the push-forward of μ).

The support X of μ a self-similar set: the unique non-empty compact set satisfying

$$X = \bigcup_{i=1}^{k} f_i(X)$$

Iterative construction



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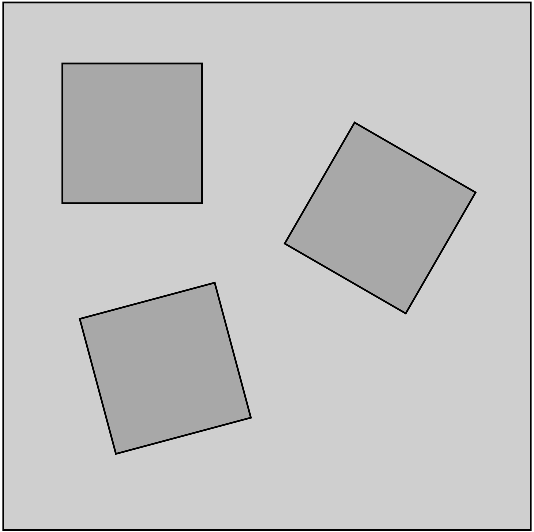

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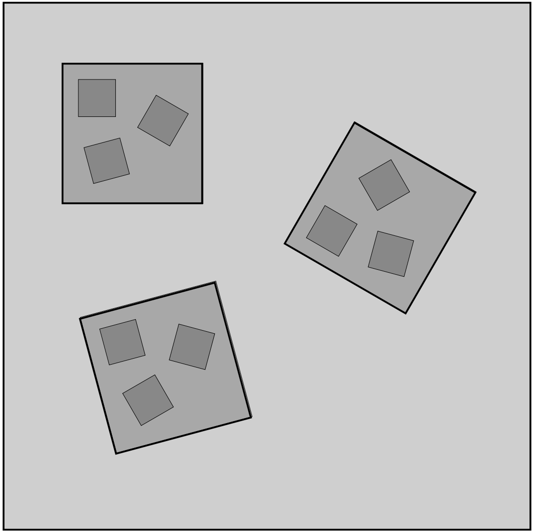

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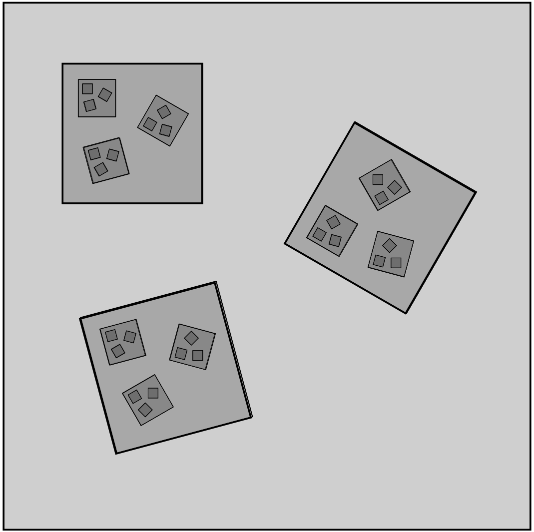

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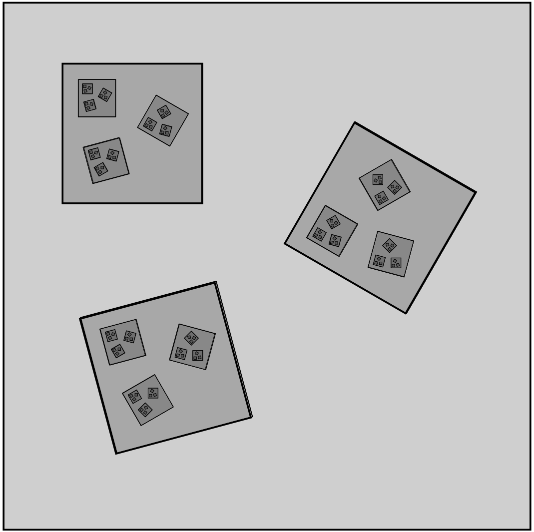

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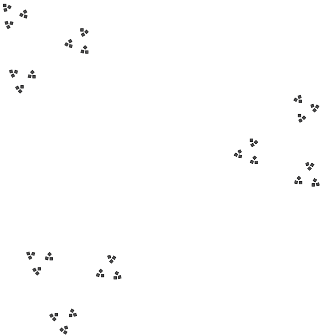

Main problem

What is the Hausdorff dimension of X?

What is the Hausdorff dimension of μ, defined as

$$\dim \mu = \alpha \iff \mu(B_r(x)) = r^{\alpha+o(1)} \text{ for } \mu\text{-a.e. } x$$

When is μ smooth? (and how smooth is it?)

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"Equidistribution conjecture"

"The random walk should spread out as much as possible (μ should be as smooth as possible), unless algebraic obstructions occur."

Classical answer [Hutchinson, Moran,...]:

Let r_i =contraction ratio of f_i. The similarity dimension is the solution s to

$$\sum r_i^s = 1$$

Theorem [Hutchinson]. One always has

$$\dim(X) \leq s$$

And, assuming the union $X = \bigcup_{i=1}^k f_i(X)$ is disjoint ("strong separation"),

$$\dim(X) = s$$

An analogous statement holds for measures.

The upper bound

By Hutchinson's theorem:

$$\dim X \leq s$$

Also: $X \subseteq \mathbb{R}^d$, therefore

$$\dim X \leq d$$

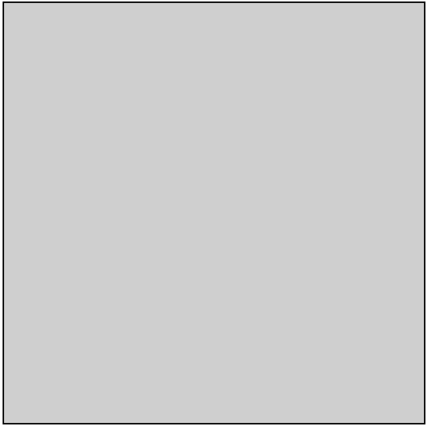
So we always have

$$\dim(X) \leq \min\{s, d\}$$

We call this the trivial upper bound.
If the inequality is strict we say there is dimension drop.

Where does s come from?

Assume 3 maps with contraction $1/4 \implies s = \frac{\log 3}{\log 4}$.



1 cube of side length 1

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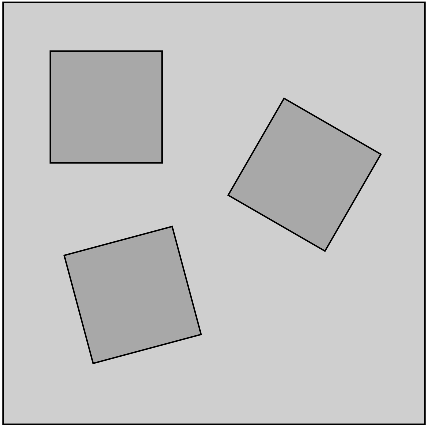

3 cubes of side length $\frac{1}{4}$

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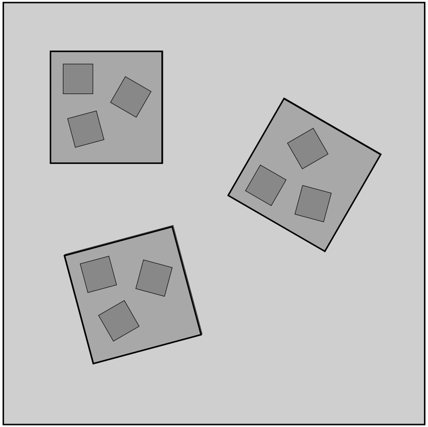

3^2 cubes of side length $\frac{1}{4^2}$

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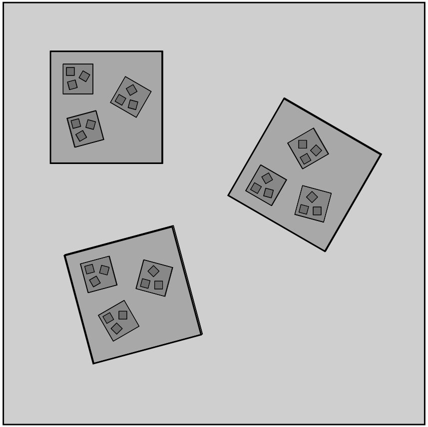

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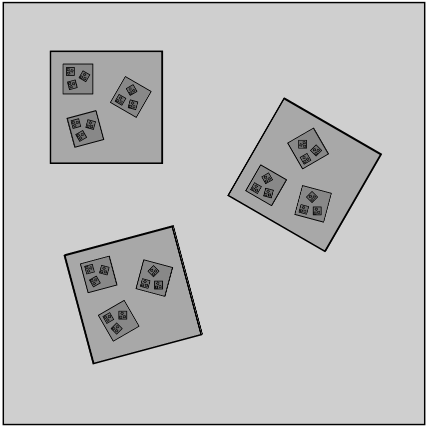

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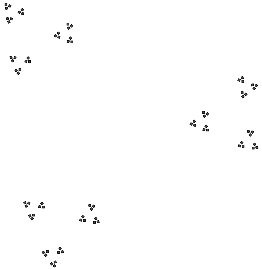
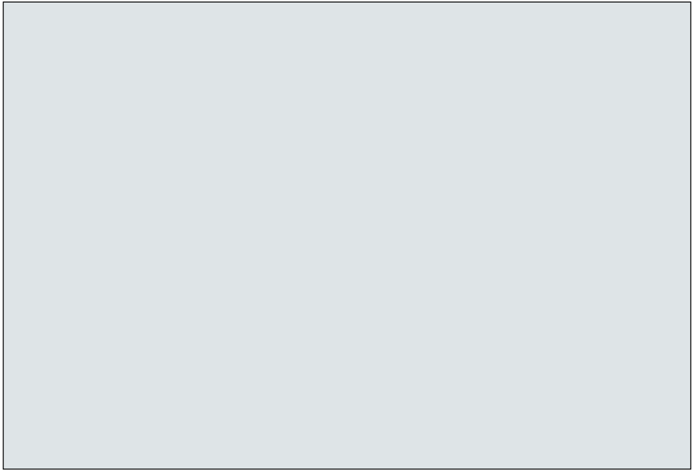


Need 3^n cubes of side length $\frac{1}{4^n}$ to cover n-th iterate $\implies \dim X = \frac{\log 3}{\log 4} = s$

Why is it only an upper bound?



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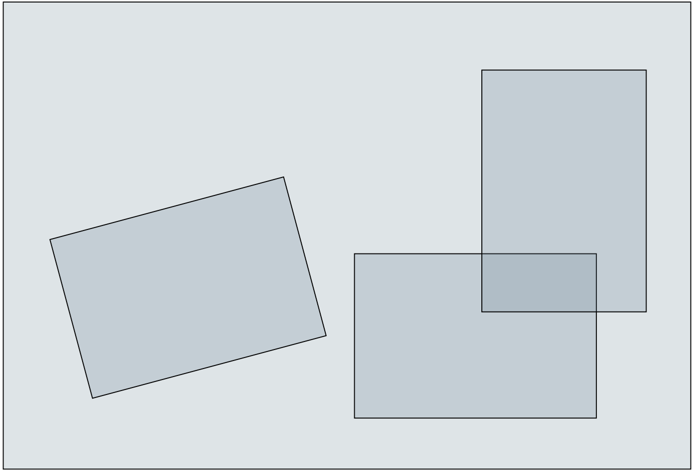

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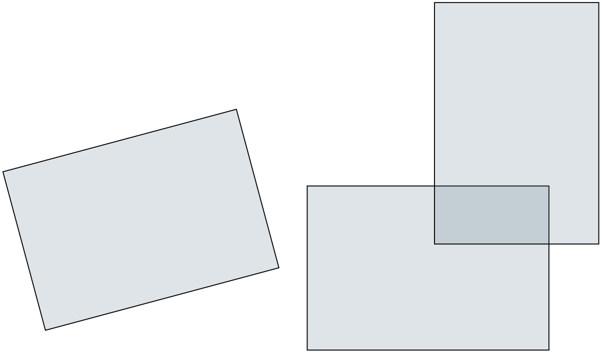

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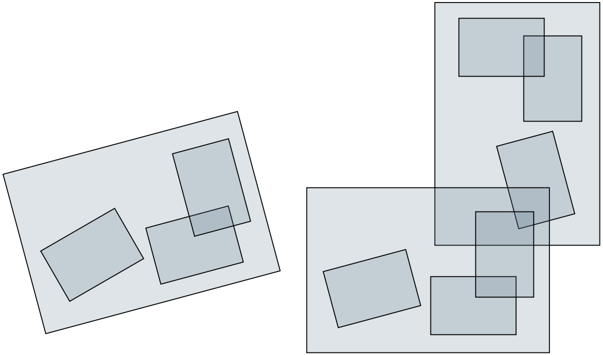

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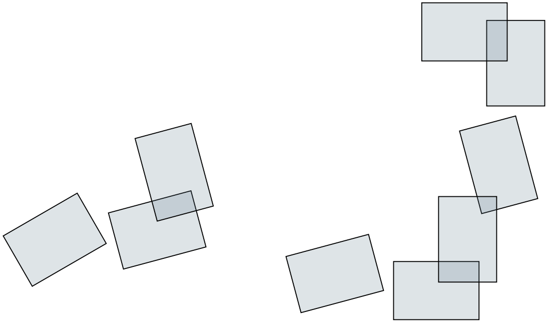

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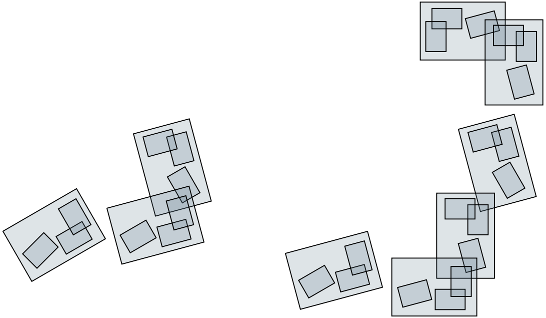

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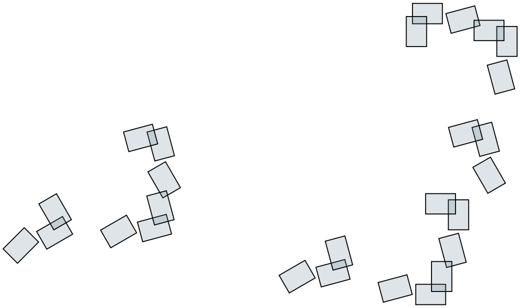

Where there are overlaps, more efficient covers may be possible.

One reason for dimension drop

For $\mathbf{i} = i_1 \dots i_n \in \{1, \dots, k\}^n$,

$$f_{\mathbf{i}} \quad = \quad f_{i_1} \circ \dots \circ f_{i_n}$$

Exact overlaps occur if there are $\mathbf{i} \neq \mathbf{j}$ with $f_{\mathbf{i}} = f_{\mathbf{j}}$.

$\iff$ $\{f_i\}_{i=1}^k$ do not freely generate the semigroup $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \{1,\dots,k\}^n}$.

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Conjecture [Erdős, Furstenberg, Simon,...]

In dimension $d = 1$, exact overlaps are the only cause for dimension drop.

Some support from analysis of parametric families (Pollicott, Simon, Solomyak, ...)

Measuring separation

Choose a left invariant Riemannian metric $d(\cdot,\cdot)$ on the similarity group (many other metrics are be equally good).

Define

$$\Delta_n \;=\; \min\left\{ d(f_{\mathbf{i}}, f_{\mathbf{j}}) \;:\; \mathbf{i},\mathbf{j} \in \{1,\ldots,k\}^n \;,\; \mathbf{i} \neq \mathbf{j} \right\}$$

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Say that $\{f_1,\ldots,f_k\}$ are exponentially separated if there exists $\rho > 0$ such that $\Delta_n \geq \rho^n$. (This property does not depend very much on the metric we choose)

Theorem (H. 2014)

In dimension $d = 1$, (at least) one of the following holds:

- $\dim X = \min\{1, s\}$
- $\Delta_n \to 0$ super-exponentially.

Equivalently: exponential separation $\implies$ no dimension drop.

An analogous statement holds for measures.

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$\implies$ Either the distance is 0, or it is at least ρ^n in absolute value, for some $\rho > 0$ depending on $f_1, \ldots, f_k$ (classical Diophantine approximation).

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$\implies$ By the theorem, there is no dimension drop.

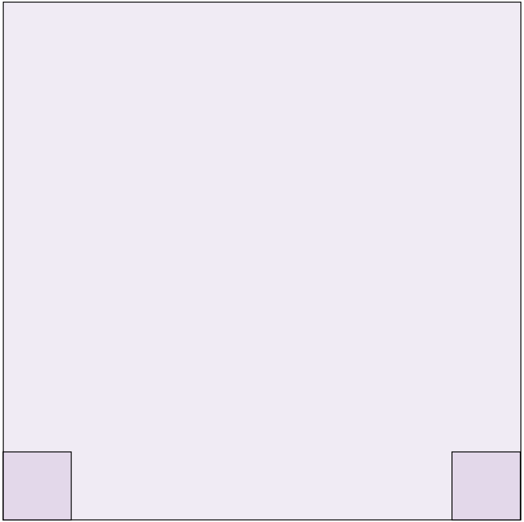
Higher dimensions

The analogous statements are false for $d > 1$: we can have

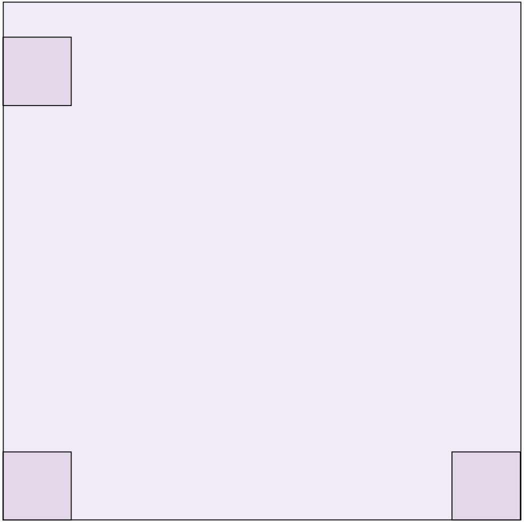
$$\dim X < \min\{s, d\}$$

but no exact overlaps (and even exponential separation).

The "excess dimension" is absorbed on vertical lines, which are invariant (under the orthogonal parts of the f_i)



Contraction $r \ll 1$



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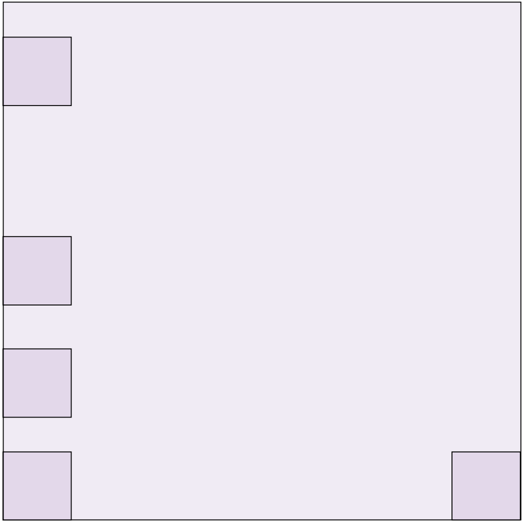

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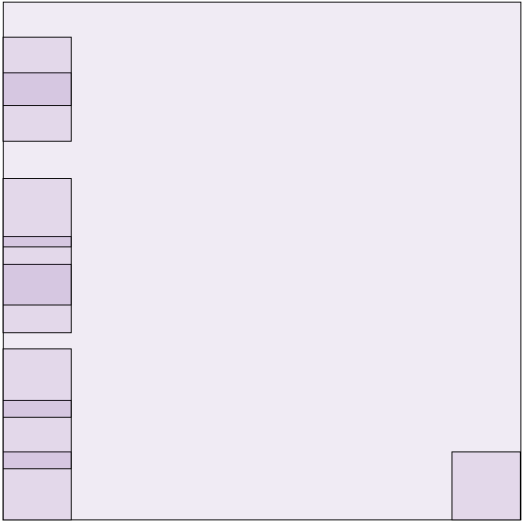

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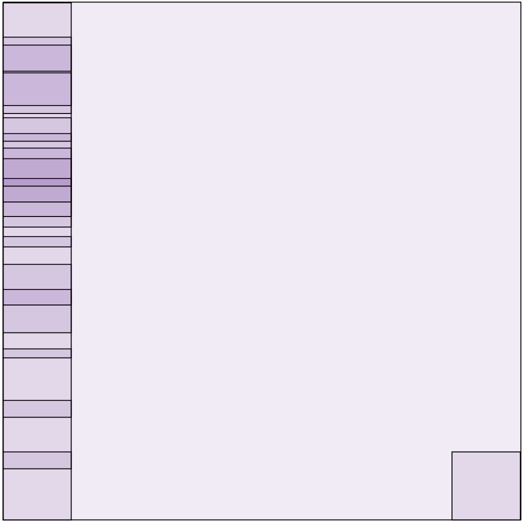

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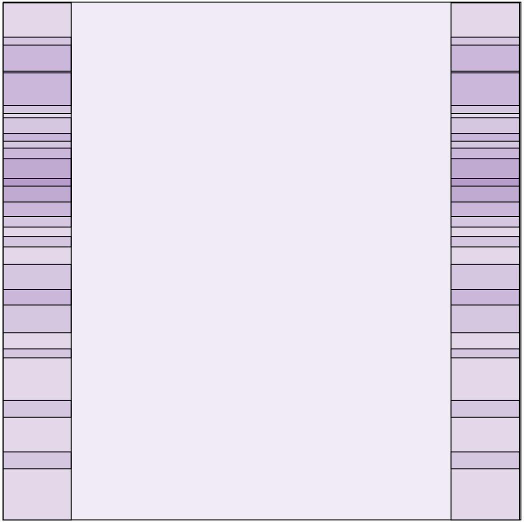

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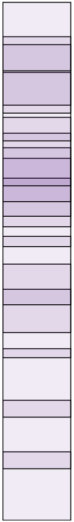

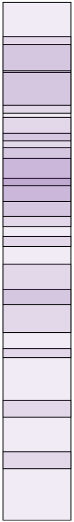

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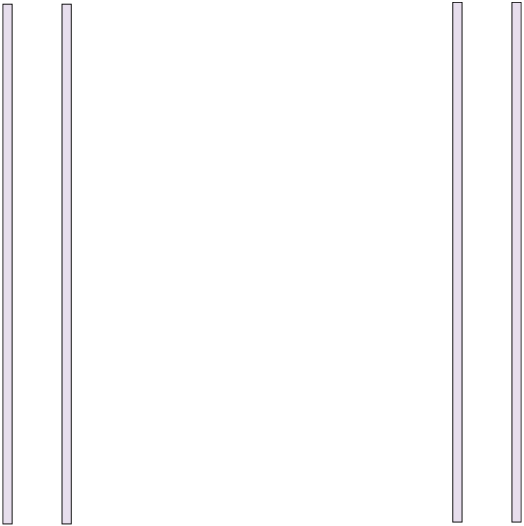

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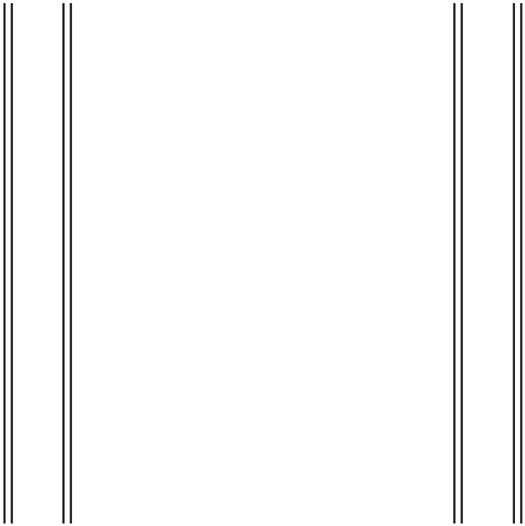

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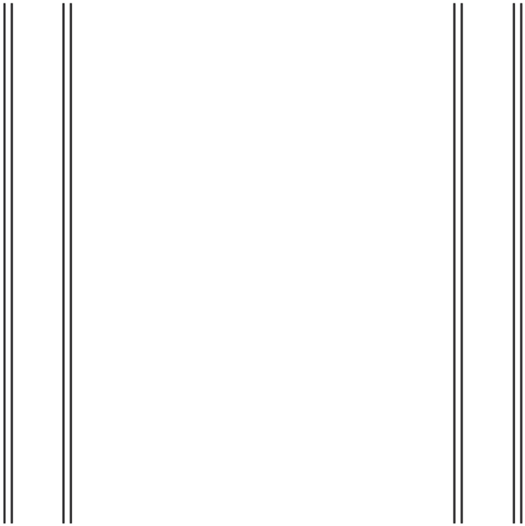

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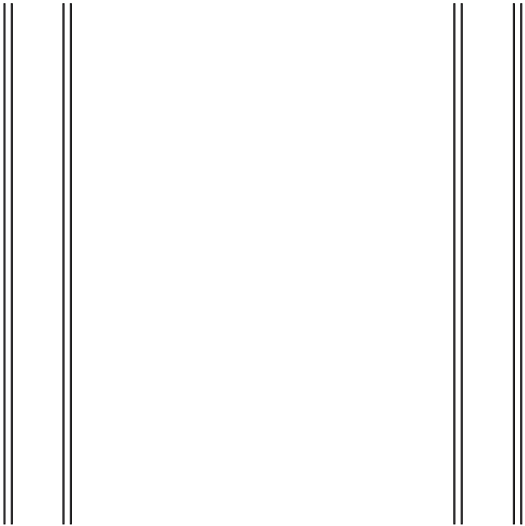
$$\dim X_1 = \frac{\log 2}{\log(1/r)} \ll 1$$



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$$\begin{aligned} \dim X_1 &= \frac{\log 2}{\log(1/r)} \\ &\ll 1 \end{aligned}$$

$$\begin{aligned} \dim X &= \dim X_1 + 1 \\ &\ll \min\{2, s\} \end{aligned}$$

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Conjecture for $d > 1$

At least one of the following holds:

- $\dim X = \min\{s, d\}$.
- Exact overlaps occur.
- $\exists \{U_i\}$-invariant subspace $0 \neq V < \mathbb{R}^d$ and $x \in X$ s.t.

$$\dim((V + x) \cap X) = \dim V$$

Theorem (H. 2014/2019?)

At least one of the following holds:

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(**Corollary**: if $\{U_i\}$ acts irreducibly, exponential separation implies no dimension drop).

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There is an analog for measures.

Compare with work of Lindenstrauss-Varjú: If $\{U_i\}$ have spectral gap and f_i have weak enough contraction, and p has high enough entropy, then μ is absolutely continuous.

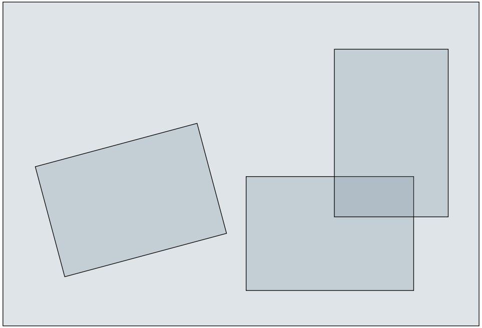
One proof ingredient: Additive combinatorics

Assume dimension drop and go to a small scale.

Locally, we see exponentially many copies - a convolution of the original measure with a "high entropy" discrete measure (this assumes exp. separation!).

Use results on entropy growth of convolutions.

One proof ingredient: Additive combinatorics



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Locally, we see exponentially many copies - a convolution of the original measure with a "high entropy" discrete measure (this assumes exp. separation!).

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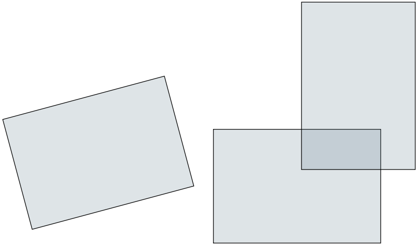

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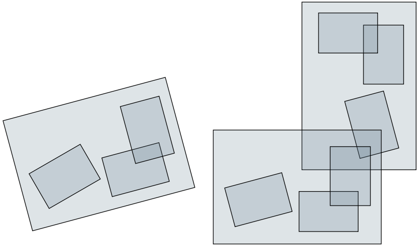

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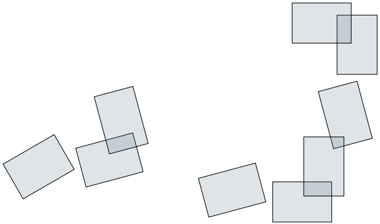

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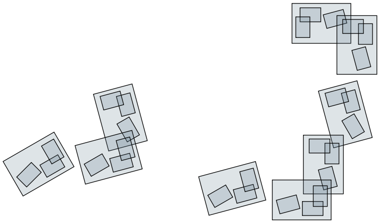

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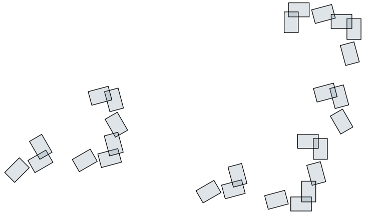

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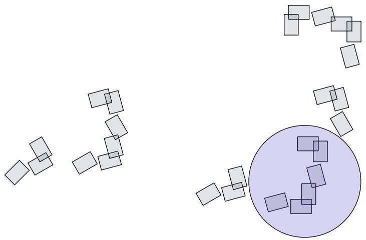

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Application: Bernoulli convolutions

Fix $0 < \lambda < 1$.

Let ν_λ = the distribution of the random real number

$$\sum_{k=0}^{\infty} \pm\lambda^k \text{ (signs i.i.d. and uniform)}$$

Then

$$\sum_{k=0}^{\infty} \pm\lambda^k = \pm 1 + \lambda \cdot \sum_{k=0}^{\infty} \pm\lambda^k$$

The inner sum has the same distribution as ν_λ, so, conditioning on the first sign, we just have ν_λ translated by ± 1 and scaled by λ.

This shows that ν_λ is self-similar.

The trivial upper bound on the dimension (analogous to that for sets) is

$$\dim \nu_\lambda \;\; \leq \;\; \min\{1, \frac{\log 2}{\log(1/\lambda)}\} = 1$$

when $1/2 \leq \lambda < 1$. So potentially there is equality, and perhaps smoothness.

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Problem

.

For which λ is ν_λ absolutely continuous?
For which λ is $\dim \nu_\lambda = 1$?

(the former implies the latter).

A very brief history:

- Jessen and Wintner 1935: Laws of pure type.
- Erdös 1939: If λ^{-1} is Pisot, then $\nu_\lambda \perp \mathrm{Leb}$ (and $\dim \nu_\lambda < 1$ [Garsia 1963]).
- Salem 1963: $\widehat{\nu}_\lambda(k) \underset{k\to\infty}{\longrightarrow} 0 \iff \lambda^{-1}$ is Pisot.
- Solomyak 1995: $\nu_\lambda \ll \mathrm{Leb}$ for a.e. $\lambda \in (1/2, 1)$.
- Peres-Schlag 2000: $\dim\{1-t < \lambda < 1 : \nu_\lambda \perp Leb\} \to 0$ as $t \to 1$.

Theorem (H. 2014)

$$\dim\{\lambda \in [\frac{1}{2}, 1) : \dim \nu_\lambda < 1\} \;=\; 0$$

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Theorem (Shmerkin 2014)

$$\dim\{\lambda \in [\frac{1}{2},1) : \nu_\lambda \ll Leb\} \;=\; 0$$

The proofs use that outside a dimension 0 set of λ, there is exponential separation. The second theorem also involves a Fourier-theoretic argument.

Theorem (Varjú 2019?)

For every $\varepsilon > 0$ there exists a $c > 0$, with the following property: If λ is an algebraic number with Mahler measure M_λ and satisfying

$$\lambda > 1 - c\min\{\log M_\lambda, (\log M_\lambda)^{-1-\varepsilon}\}$$

Then $\nu_\lambda \ll \mathrm{Leb}$.

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Theorem (Breuillard-Varjú 2019?)

$$\{\lambda \in [\frac{1}{2}, 1) : \dim \nu_\lambda < 1\} = \overline{\{\lambda \in [\frac{1}{2}, 1) : \dim \nu_\lambda < 1 \text{ and } \lambda \text{ is algebraic}\}}$$

The proofs rely partly on more quantitative estimates of what exponential separation.

Related results using similar philosophy:

1. Shmerkin-Solomyak (2015,2016): Improved results on smoothness of projections of self-similar measures.

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5. H.-Rapaport (in progress): Replace separation with exponential separation for self-affine sets in $\mathbb{R}^2$. New phenomenon: In reducible case, must exclude set supported on quadratic curves! (Examples exist by Bandt-Kravchenko).

The main conjecture is still open...