

Ruelle resonances for pseudo-Anosov maps

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- $T : X \rightarrow X$ a continuous map
- m a probability measure on X , preserved by T

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Goal: speed of convergence? Asymptotic expansion?

The doubling map

$T(x) = 2x \pmod{1}$ on the circle \mathbb{S}^1

m : Lebesgue measure

Theorem

Assume $f, g \in C^\infty(\mathbb{S}^1)$. Then, for any $\epsilon > 0$,

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Proof.

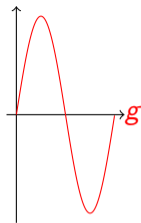
$$g(x) = \sum \hat{g}_k e^{2i\pi kx}.$$

$$g(T^n x) = \sum \hat{g}_k e^{2i\pi k 2^n x}.$$

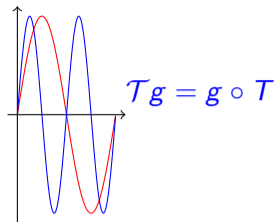
$$\int f \cdot g \circ T^n \, dm = \hat{f}_0 \overline{\hat{g}_0} + \sum_{k \neq 0} \hat{f}_{2^n k} \overline{\hat{g}_k}.$$

The Fourier coefficients of f decay faster than any polynomial. □

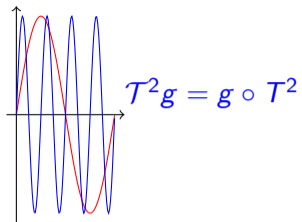
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 $\mathcal{T}g = g \circ T.$



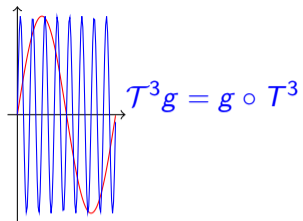
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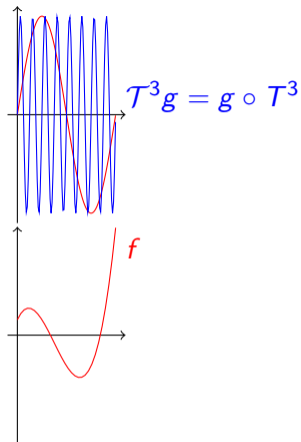


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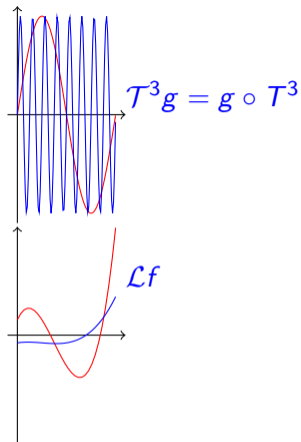
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Transfer operator $\mathcal{L} = \mathcal{T}^*$:
 $\int f \cdot g \circ T \, dm = \int \mathcal{L}f \cdot g \, dm$
 $\mathcal{L}f(x) = \frac{1}{2}(f(\frac{x}{2}) + f(\frac{x+1}{2}))$.



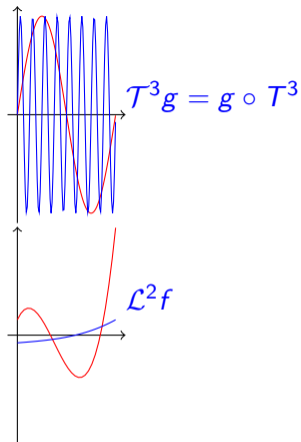
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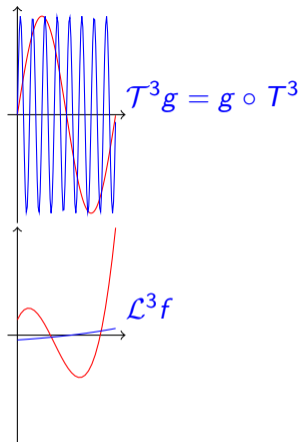
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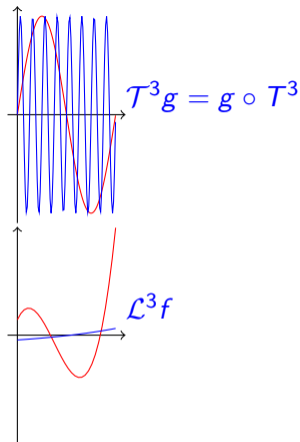


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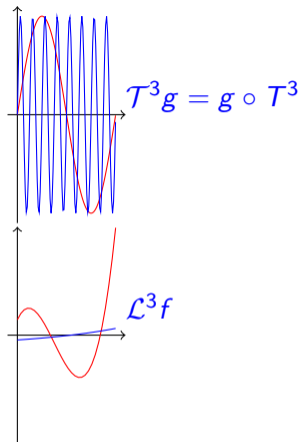


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If f is C^∞ and $\int f \, dm = 0$, then $\mathcal{L}^n f$ tends to 0 superexponentially fast in C^∞ .

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 If $\int g \, dm = 0$, then $\mathcal{T}^n g$ tends to 0 superexponentially fast as a distribution.

Uniformly expanding maps of the circle

$T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ a C^∞ map, with $T' \geq \beta > 1$.
 m Lebesgue measure. Assume T preserves m .

Theorem

Let $f, g \in C^\infty$. For all $\epsilon > 0$, there is an expansion

$$\int f \cdot g \circ T^n \, dm = \sum_{|\lambda_i| \geq \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n),$$

where

- λ_i is a sequence tending to 0 (the Ruelle resonances).
- $c_i(f, g)$ coefficients depending on f and g .

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Just like for powers of matrices, there could be polynomial terms. We ignore them for simplicity.

Sketch of proof, using the transfer operator $\mathcal{L} = \mathcal{T}^*$

Let $C \geq 0$. Decompose a function f as

$$f = \sum_{|k| \leq C} \hat{f}_k e^{2i\pi kx} + \sum_{|k| > C} \hat{f}_k e^{2i\pi kx} = f_{\leq C} + f_{> C}.$$

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$$\mathcal{L}f = \mathcal{L}f_{\leq C} + \mathcal{L}f_{> C} = \mathcal{L}_{\leq C}f + \mathcal{L}_{> C}f.$$

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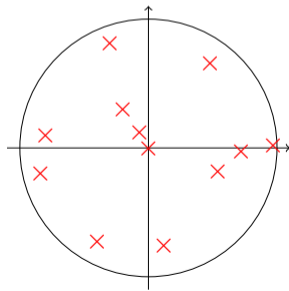
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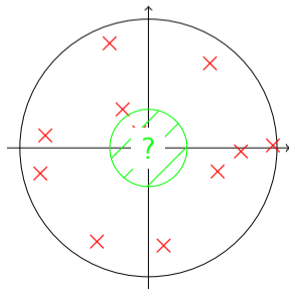
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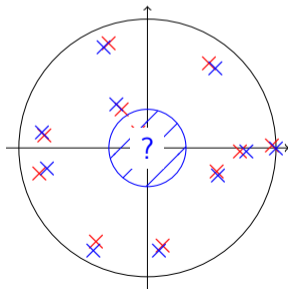
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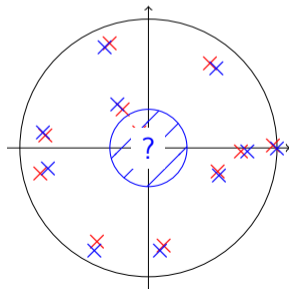
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One could also do the same with \mathcal{T} acting on $(C^r)^*$.

Ruelle resonances: general setting

Definition

A probability-preserving system (X, T, m) has a Ruelle spectrum $\{\lambda_i\}$ if, for all C^∞ functions f and g , for all $\epsilon > 0$,

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Further questions:

- Is the Ruelle spectrum trivial (like for $x \mapsto 2x \pmod{1}$)?
- Asymptotics of the number of eigenvalues with $|\lambda_i| \geq \epsilon$
- Speed of decay of correlations (i.e., gap between $\lambda_0 = 1$ and $|\lambda_1|$)
- Compute the Ruelle spectrum in some examples

Ruelle resonances for linear pseudo-Anosov maps

X : compact connected surface, genus g

T : linear pseudo-Anosov map, preserving orientations

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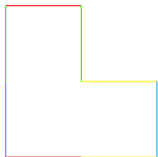
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Example

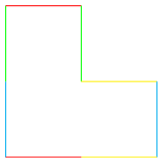
$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ the cat map, acting on \mathbb{T}^2 .

Its Ruelle spectrum is $\{1\}$.

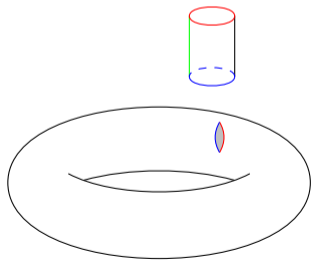
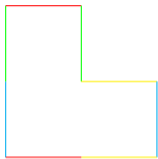
Constructing a translation surface



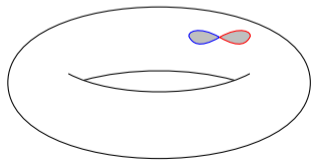
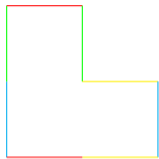
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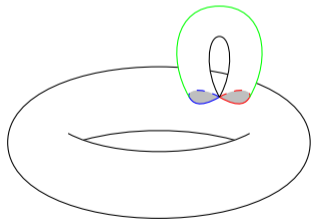
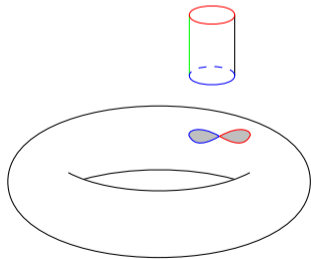
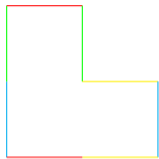
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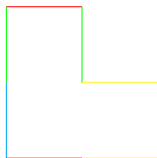
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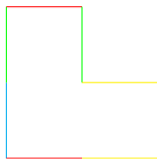
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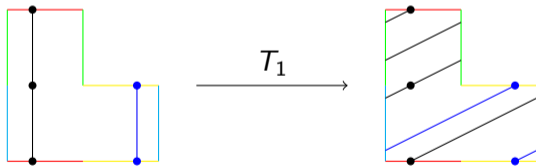
Constructing a pseudo-Anosov map



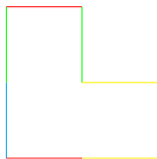
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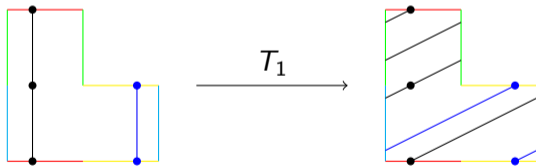
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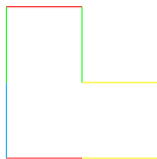


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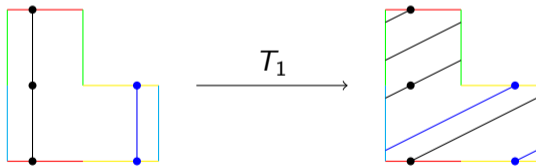


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Compose the above two: one gets a map $T : X \rightarrow X$ locally given by the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$.
This is a linear pseudo-Anosov map.

T linear pseudo-Anosov map, with expansion factor $\beta > 1$ and spectrum on H^1 :
 $\{\beta, \beta^{-1}, \mu_1, \dots, \mu_{2g-2}\}$.

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Steps of the proof:

- 1 Ruelle resonances make sense
- 2 Cohomology gives rise to Ruelle resonances
- 3 All Ruelle resonances come from cohomology

Ruelle resonances make sense

Goal: construct a Banach space on which $f \mapsto f \circ T$ has a small essential spectral radius.

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For contracting maps: smooth functions

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In hyperbolic cases, anisotropic space

- smooth in the contracting (vertical) direction
- dual of smooth in the expanding (horizontal) direction

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Smoothness indices k_h and k_v . Let L_v be the vertical derivative.

$$\|f\| = \sup_{i \leq k_v} \sup_{\substack{I \text{ horizontal interval} \\ |I|=1, I \cap \Sigma = \emptyset}} \sup_{\phi \in C_c^{k_h}(I), \|\phi\|_{C^{k_h}} \leq 1} \int_I \phi \cdot L_v^i f \, dx$$

\mathcal{B} = completion of $C_c^\infty(M - \Sigma)$ for $\|\cdot\|$.



Cohomology gives rise to Ruelle resonances

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To show that $\beta^{-n}\mu$, $n > 1$, is also a Ruelle resonance, use $L_h^{n-1}f$ where f eigenfunction for $\beta^{-1}\mu$.

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All Ruelle resonances come from cohomology

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Consider α with $E_{\alpha} \neq 0$. We want to show that $\alpha = 1$, or α is of the form $\beta^{-n}\mu_i$.

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- If $[g] = 0$, then $g = L_h g_1$, with $g_1 \in E_{\beta^{k+1}\alpha} \cap \ker L_v$. Repeat, until $[g_{\ell}] \neq 0$. Then $\beta^{k+\ell+1}$ is an eigenvalue of T^* . □