Geometry of Teichmüller curves

Martin Möller

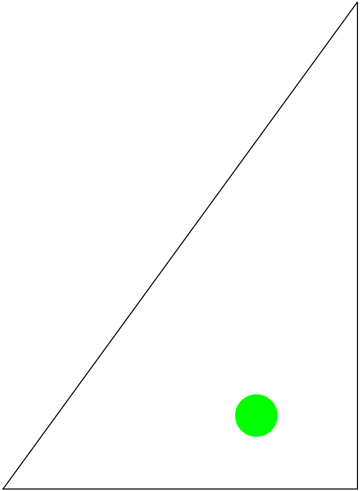
02. August 2018

Briefly, in the language of complex geometry:

A **Teichmüller curve** is an algebraic curve $C \to \mathcal{M}_g$ immersed in the moduli space of curves which is totally geodesic for the Teichmüller metric.

The **Teichmüller metric** coincides with the **Kobayashi metric**.

Motivation: dynamics on rational polygonal billiards:



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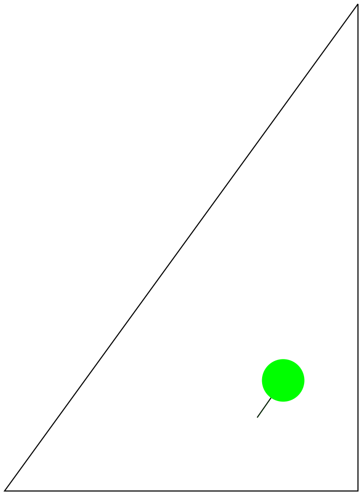

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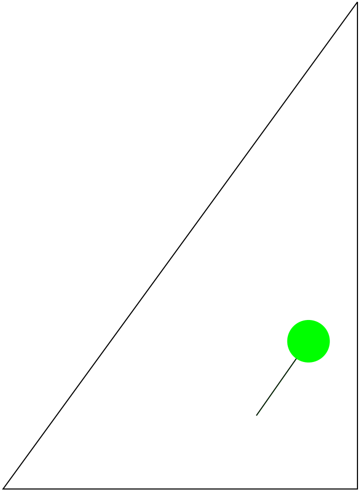

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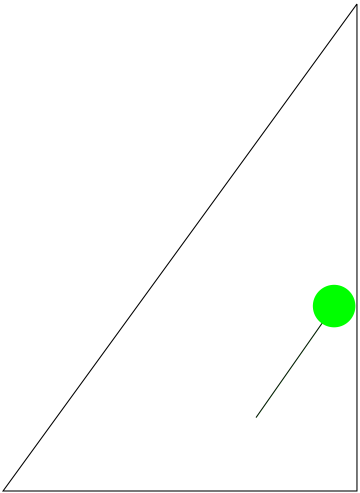

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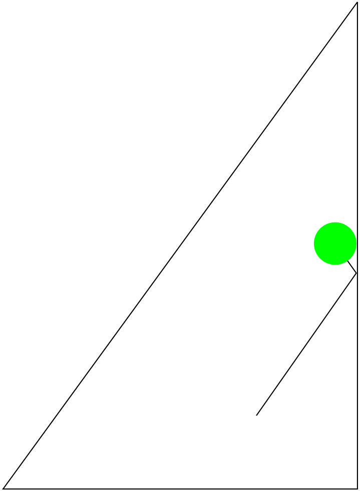

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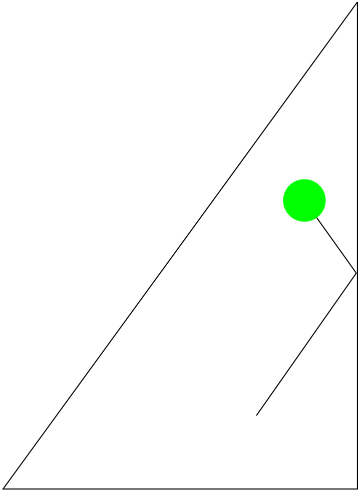

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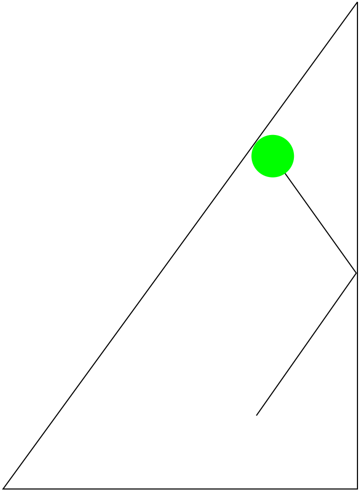

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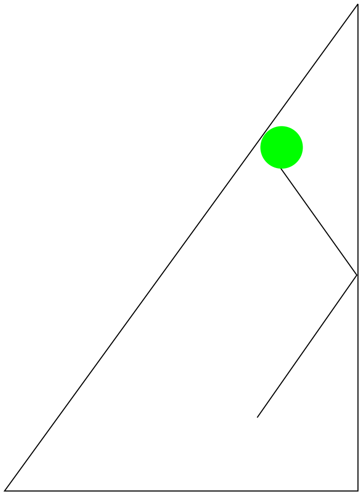

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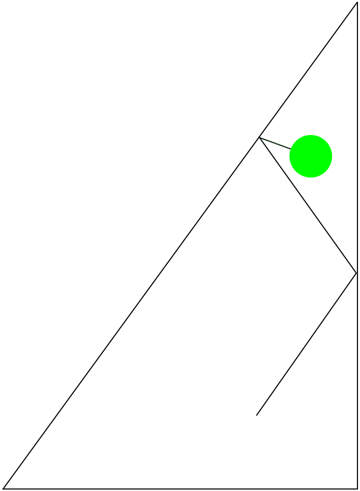

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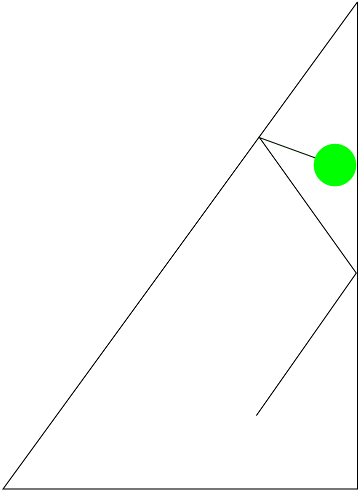

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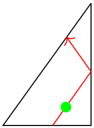

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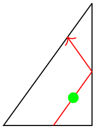

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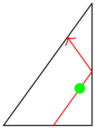

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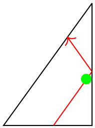

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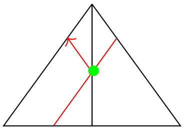
Reflect the polygon instead of reflecting the path!



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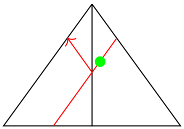

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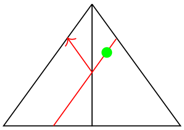

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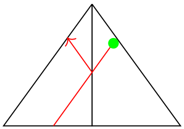

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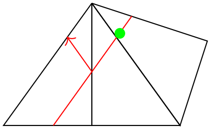

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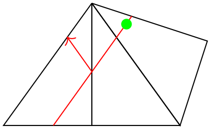

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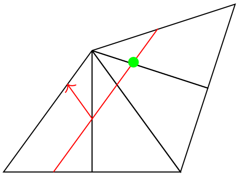

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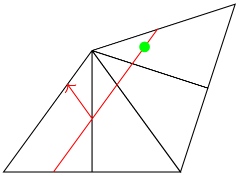

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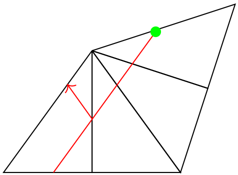

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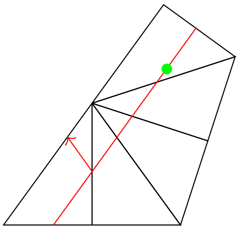

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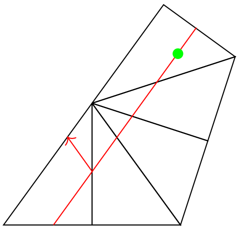

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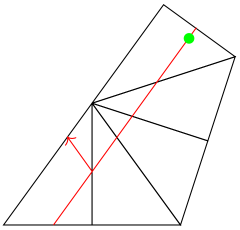

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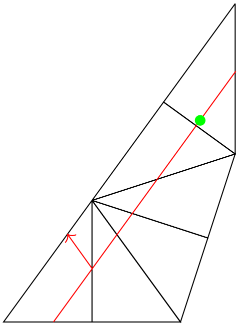

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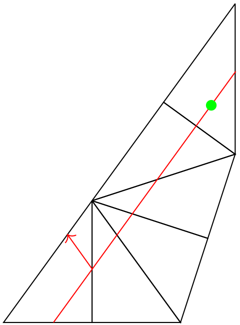

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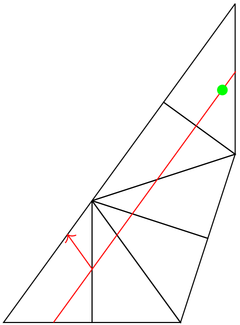

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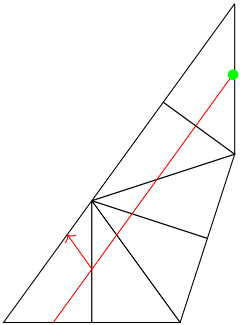

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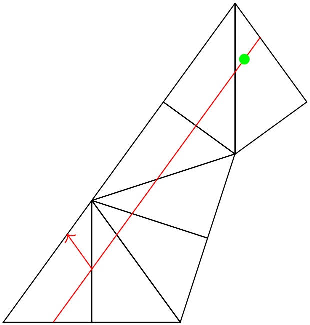

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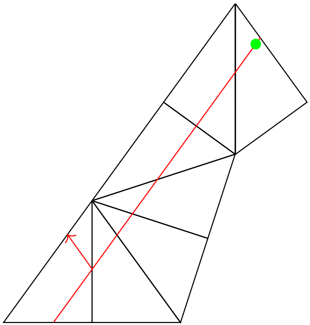

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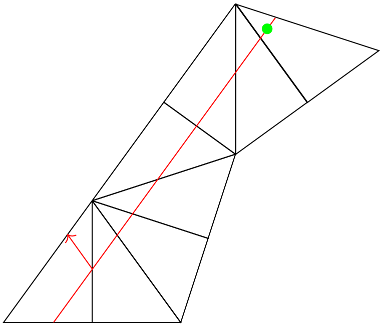

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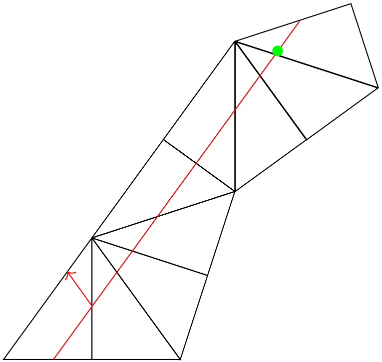

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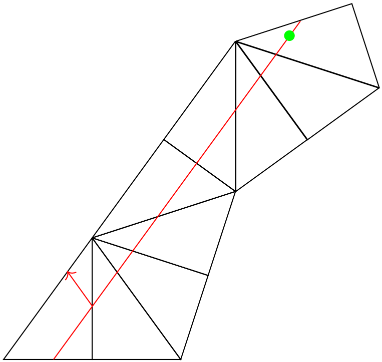

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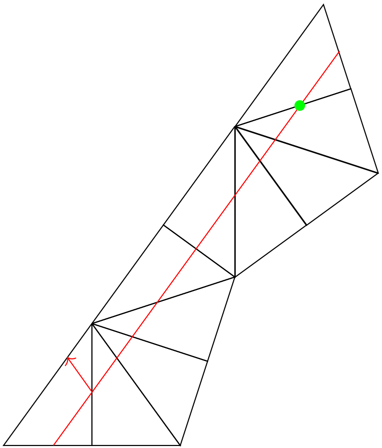

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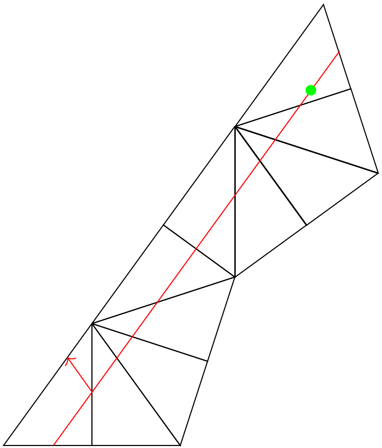

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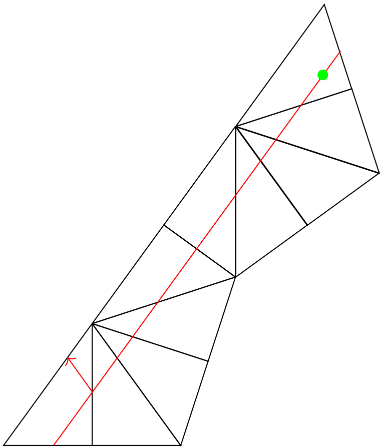

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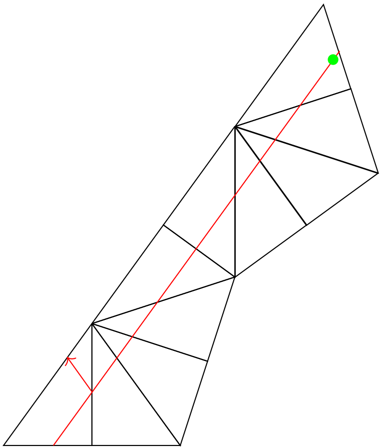

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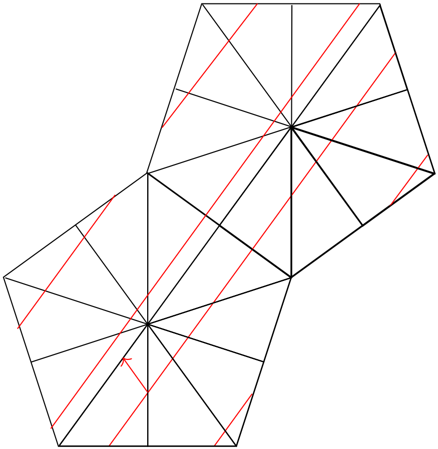
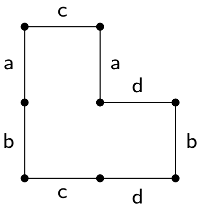


A **flat surface** (X, ω) is a compact Riemann surface X together with a non-zero holomorphic one-form ω.

Example: A flat surface of genus $g(X) = 2$ with a conical singularity of angle 6π, i.e. where ω has a zero of order 2.



On the **unfolding of a rectangle**, the straight line flow satisfies the following dichotomy:

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"Trajectories are never dense in **part** of the surface"

The dichotomy

- The trajectories in the direction Θ are periodic (or end in a singularity),
- The trajectories in the direction Θ are uniformly distributed (or end in a singularity)

also holds for the unfolding of the $(\pi/2, \pi/5, 3\pi/10)$-triangle!

Such flat surfaces are called **dynamically optimal**.

Dynamics on flat surfaces (X, ω)

is determined by and determines

Geometry of algebraic curves in the moduli space $\mathcal{M}_g$.

The moduli space of flat surfaces $\Omega\mathcal{M}_g$ is a vector bundle over the moduli space of curves $\mathcal{M}_g$ (with the zero section removed).

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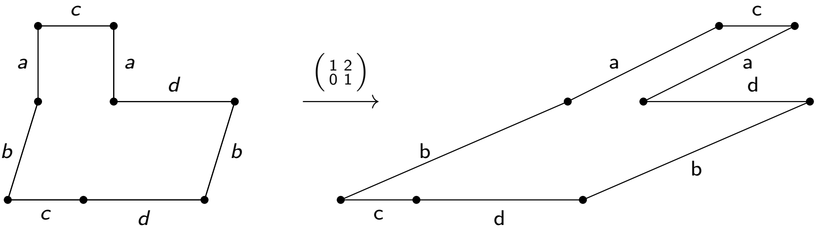
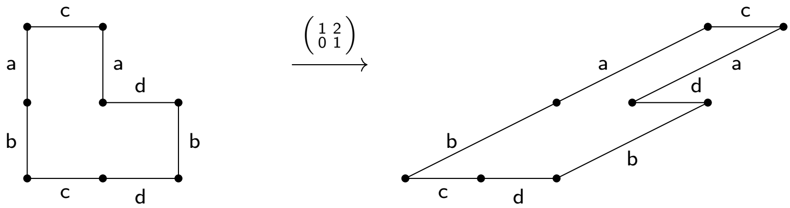
Strata. The moduli space $\Omega\mathcal{M}_g$ is stratified according to the multiplicities of the zeros:

$$\Omega\mathcal{M}_g = \bigcup_{\mathbf{m} \vdash 2g-2} \Omega\mathcal{M}_g(\mathbf{m}), \quad \mathbf{m} = (m_1, \ldots, m_n)$$

$\mathrm{SL}_2(\mathbb{R})$ acts on flat surfaces:



Sometimes elements of $\mathrm{SL}_2(\mathbb{R})$ preserve a flat surface. For example:



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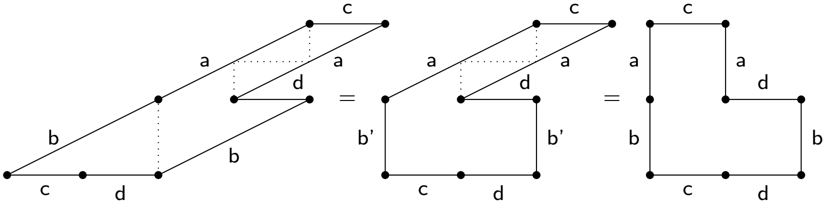

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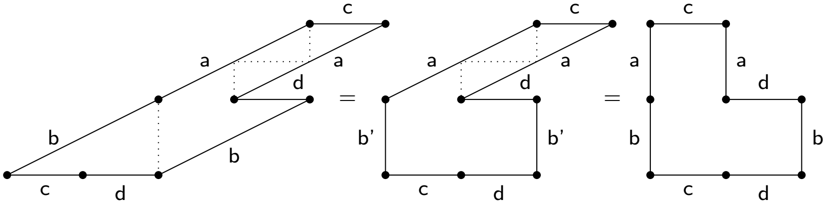

The **Veech group** $\mathrm{SL}(X,\omega)$ consists of those elements in $\mathrm{SL}_2(\mathbb{R})$ that stabilize a flat surface (X,ω).

- In the *L*-shaped triple cover example the Veech group is index two in $\Delta(2,3,\infty) = \mathrm{SL}_2(\mathbb{Z})$.

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- In the double pentagon example the Veech group is the triangle group $\Delta(2, 5, \infty)$.

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- The converse is true in genus two, but not in higher genus.
  (McMullen 2004, Smillie-Weiss 2006)
- The image of the orbit $\mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega)$ of a Veech surface in $\mathcal{M}_g$ is an algebraic curve.

  These curves are **Teichmüller curves**.

Geometry and dynamics of the flat (Veech) surface:

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- Construction and classification of Veech surfaces
- Asymptotic growth rate of the number of closed trajectories of bounded length?
- (Can you hear billiard tables? Determine it by the spectrum of geodesics or bouncing sequences!
  Duchin-Erlandsson-Leininger-Sadanand 2018;
  Calderon-Coles-Duchin-Lanier-Oliveira 2018)

Geometry of the Teichmüller curves:

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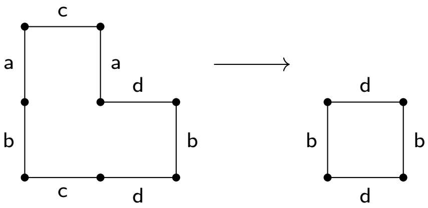
- Construction and classification of Teichmüller curves?
- Euler characteristic? Number of cusps? Elliptic fixed points?

Finiteness: Bainbridge-M. 2012/2016, Matheus-Wright 2015, Eskin-Filip-Wright 2017
Geometry: Bainbridge 2007, McMullen 2005, Lanneau-Nguyen 2014, Zagier-M. 2015, M.-Torres 2018, Mukamel 2014, Torres-Zachhuber 2015

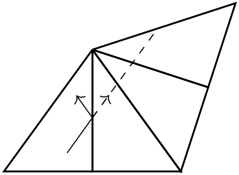
The following Veech surface arises from a covering construction:

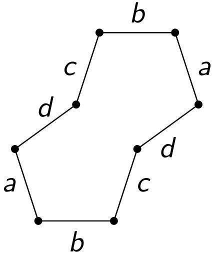


These are called **imprimitive**.

We focus on **primitive** Veech surfaces, aiming for their classification.

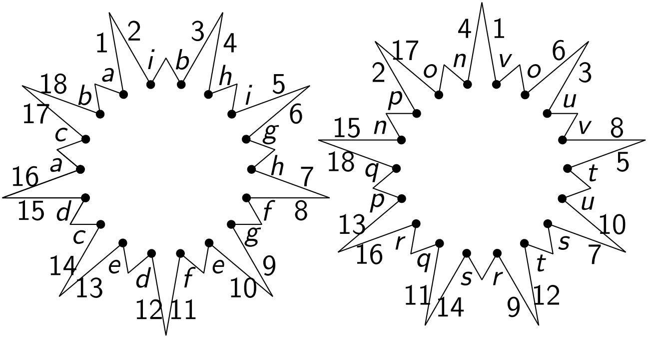
Veech (1989), Ward (1998):
Unfoldings of the $(\pi/n, \pi/n, (n-2)\pi/n)$-triangles and the $(\pi/2n, \pi/n, (2n-3)\pi/2n)$-triangles are Veech surfaces.





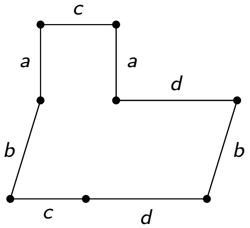
The only Veech surfaces discovered by computing the Veech group!

Semiregular polygons whose Veech groups are triangle groups (Bouw-M., 2007)



Method: Characterization of Teichmüller curves by their variation of Hodge structures.

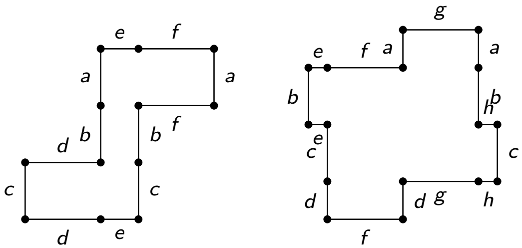
The series of L-shaped tables $\Omega\mathcal{M}_2(2)$
(Calta 2003; McMullen 2003)



$$a = (0, \lambda)$$
$$b = (h, t)$$
$$c = (\lambda, 0)$$
$$c + d = (w, 0)$$
$$\lambda = (e + \sqrt{D})/2$$

Method: SL_2-invariance of real multiplication by quadratic fields.

The series of Prym examples in $\Omega\mathcal{M}_3(4)$ and $\Omega\mathcal{M}_4(6)$ (McMullen, 2007)



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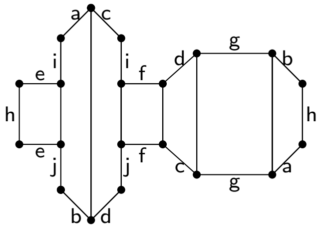
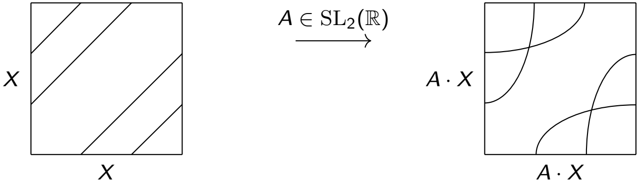
The series of Gothic examples in $\Omega\mathcal{M}_4(2,2,2)$
(Mukamel-McMullen-Wright, 2017)



Method: Cut out by real-linear equations in period coordinates.

$\mathrm{SL}_2(\mathbb{R})$ destroys symmetries, at best correspondences remain.



Correspondences on Y give endomorphisms of $\mathrm{Jac}\,(Y)$.

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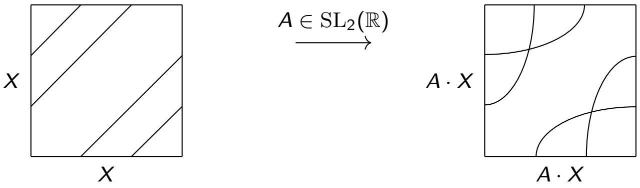


Jacobians of Veech surfaces admit real multiplication (M., 2005). Teichmüller curves map to the locus of abelian varieties with real multiplication, i.e. to Hilbert modular varieties.

Infinite families of Teichmüller curves live on Hilbert modular surfaces (Eskin-Filip-Wright 2017).

The **Hilbert modular surface** for $K = \mathbb{Q}(\sqrt{D})$ is

$$X_K \quad = \quad \mathrm{SL}_2(\mathcal{O}_K) \setminus \mathbb{H}^2_{\mathbb{C}}$$

Hilbert modular surfaces parametrize abelian surfaces with real multiplication.

The **Hilbert modular surface** for the order $\mathcal{O}_D$ and an $\mathcal{O}_D$-ideal $\mathfrak{b}$ is

$$X_D(\mathfrak{b}) \;=\; \mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee) \setminus \mathbb{H}_{\mathbb{C}}^2$$

Hilbert modular surfaces parametrize abelian surfaces with real multiplication and a polarization of degree $N(\mathfrak{b})$.

The **Kobayashi distance** on a Hilbert modular surface is

$$d_K((x_1, y_1), (x_2, y_2)) = \max\{d_P(x_1, x_2), d_P(y_1, y_2)\}$$

Here: d_P is the Poincaré metric on $\mathbb{H}_{\mathbb{C}}$.

Teichmüller curves generated by Veech surfaces (X, ω) are also Kobayashi geodesics on Hilbert modular surfaces.

The diagonal

$$\{(\quad z, \quad\quad z),\ z \in \mathbb{H}_{\mathbb{C}}\} \subset \mathbb{H}_{\mathbb{C}}^2$$

descends to an algebraic curve in X_K.

Twisted diagonals for $M \in \mathrm{GL}_2(K)$

$$\{(M \cdot z,\, M^\sigma \cdot z),\ z \in \mathbb{H}_\mathbb{C}\} \subset \mathbb{H}_\mathbb{C}^2$$

give rise to **modular curves**
(also known as Hirzebruch-Zagier cycles, Shimura curves).

Rigidity of the Kobayashi metric (Schwarz Lemma):

Kobayashi geodesics $C \to X_D(\mathfrak{b})$ are algebraic curves that arise as images of

$$\{(z, \varphi(z)),\ z \in \mathbb{H}_\mathbb{C}\} \subset \mathbb{H}_\mathbb{C}^2$$

for some holomorphic map $z \mapsto \varphi(z)$ that is (in general) **not** the graph of a Möbius transformation.

A **Hilbert modular form** of weight (k, ℓ) is a holomorphic function $f : \mathbb{H}_{\mathbb{C}}^2 \to \mathbb{C}$ such that

$$f(\gamma\tau_1, \gamma^\sigma\tau_2) \;=\; (c\tau_1 + d)^k (c^\sigma\tau_2 + d^\sigma)^\ell f(\tau_1, \tau_2)$$

for $\gamma \in \mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee)$.

Main examples: Eisenstein series and theta functions ....
... are both of parallel weight $k = \ell$.

A **Hilbert theta function** is a function $\theta : \mathbb{C}^2 \times \mathbb{H}_{\mathbb{C}}^2 \to \mathbb{C}$, quasi-elliptic in $\mathbf{u} \in \mathbb{C}^2$ and modular in $\mathbf{z} \in \mathbb{H}_{\mathbb{C}}^2$.

The Nullwert of the derivative

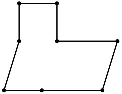
$$D_2\theta(\mathbf{z}) \;=\; \frac{\partial}{\partial u_2}\theta(\mathbf{u},\mathbf{z})\Big|_{\mathbf{u}=0}$$

is a Hilbert modular form of weight $(1/2, 3/2)$.

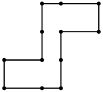
The images of the Teichmüller curves in $\Omega\mathcal{M}_2(2)$ in the Hilbert modular surfaces X_D are the vanishing locus

$$\{\mathbf{z} \in \mathbb{H}_{\mathbb{C}}^2 : \prod_{(m,m')\mathrm{odd}} D_2\theta_{m,m'}(\mathbf{z}) = 0\}$$

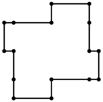
of a modular form of weight $(3,9)$
(Bainbridge 2008, M.-Zagier 2015)



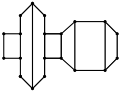
$$\prod_{(m,m')\,\mathrm{odd}} D_2\theta_{m,m'}$$



$$\begin{vmatrix} D_2\theta_0 & D_2\theta_1 \\ D_2\theta_0' & D_2\theta_1' \end{vmatrix} \quad \left({}' = \frac{\partial}{\partial z_2}\right)$$



$$\begin{vmatrix} \theta_0 & \theta_1 & \theta_2 & \theta_3 \\ \theta_0' & \theta_1' & \theta_2' & \theta_3' \\ \theta_0'' & \theta_1'' & \theta_2'' & \theta_3'' \\ \theta_0''' & \theta_1''' & \theta_2''' & \theta_3''' \end{vmatrix}$$



$$D_2\theta_0 \cdot D_2\theta_1 - D_2\theta_3 \cdot D_2\theta_4$$

(Zagier-M. 2015, M. 2014, M.-Torres 2018)

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- as the vanishing locus of certain Hilbert modular forms of non-parallel weight.