Transport in partially hyperbolic fast-slow systems

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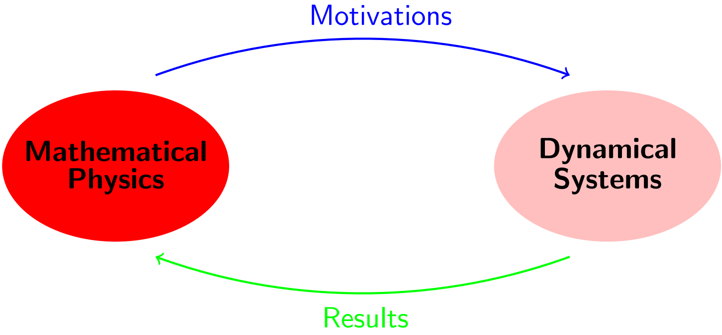
About myself



Fourier Law

I have been taught (and even taught myself ☹) the following "derivation" of the heat equation:

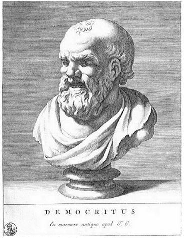
If heat u is a fluid, then it must satisfy

$$\partial_t u = \mathrm{div} j$$

where j is the current. Assume that (Fourier Law) $j = \kappa \nabla u$, then

$$\partial_t u = \mathrm{div}(\kappa \nabla u).$$

Atoms



Democritus
460–370 BC

Maxwell
1831–1879

Boltzmann
1844–1906

Einstein
1905
(from Wikipedia)

The world is made of atoms

Heat is the average local Kinetic energy per particle in a body

The "real" thing

A rigorous (classical) derivation of the heat equation must

- start from the Hamilton equations of the N particles of a body
- show that the local energy density satisfies the heat equation

and do it for $N \sim 10^{23}$!

Much ado about ....

Tremendous amount of work:

Starting with Rieder, Lebowitz, and Lieb; [J.Mat.Phys. 1967] on the harmonic crystal which found anomalous conductivity in $d < 2$.

Followed by E.G.D. Cohen, J. Bricmont, J.P. Eckmann, G. Gallavotti, A. Kupiainen, O.E. Lanford, S. Olla, E. Presutti, D. Ruelle, Ya.G. Sinai, H. Spohn, S. R. S. Varadhan, H.-T. Yau and L.-S. Young ..... just to mention a few

Randomness

The best existing results are for random microscopic dynamics: hydrodynamic limit, a field largely influenced by the work of S.R.S. Varadhan [CMP 1988, Pitman Res. Notes Math. 1993].

Randomness

Two important ingredients used in the stochastic results:

- separation of scales (*hydrodynamics limit*)
- decay of correlations (mixing and/or loss of memory) of relevant observables under the stochastic dynamics

Loss of memory, ... humm

But how can deterministic motion display loss of memory?

It seems a contradiction!

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But how can deterministic motion display loss of memory?

It seems a contradiction!

- Sensitive dependence on initial conditions (Poincarè, Smale)
- Random initial conditions (Kolmogorov)

Chaos

General belief: chaotic motion enjoys fast decay of correlations.

This was first substantiated by Sinai, Ruelle and Bowen ('70).

Vast efforts to extend such results to more general systems:

L.-S. Young, D. Dolgopyat, O. Sarig ....

Are there results for Hamiltonian flows?

Hamiltonian dynamics

Exponential decay of correlations for Hölder observables in

- Geodesic flows in negative curvature

  (Chernov, Dolgopyat, L. [Annals 2004])

- Hyperbolic Billiards

  (Sinai $\longrightarrow$ Baladi-Demers-L. [Invent. 2018])

Can we use these results to investigate heat transport?

We need a concrete model

My take: hot stones

Testo (taken from e-bay)

No convection !

Piadina Romagnola
(taken from giallo-zafferano)

Hot stones – formal definitions

Consider a finite region in a lattice, say $\Lambda \in \mathbb{Z}^{\nu}$. At each point of the lattice there is a mechanical system.

Assume the local system to be modelled by identical geodesic flows in negative curvature (M, ϕ_t), $\dim M = d$.

Thus at the point $z \in \mathcal{H}$ we will have the local Hamiltonian

$$\mathcal{H}(q_z, p_z) = \frac{1}{2}\langle p_z, p_z \rangle$$

$(q_x, p_x) \in TM$.

Hot stones

Consider then the total hamiltonian

$$\mathcal{H}_{\varepsilon,\Lambda}(q,p) = \sum_{z\in\Lambda} \mathcal{H}(q_z, p_z) + \varepsilon \sum_{\substack{x,z\in\Lambda \\ \|x-z\|=1}} V(q_x, q_z)$$

For $\varepsilon = 0$ the local energies are conserved quantities.

For $\varepsilon > 0$ the interaction yields an energy exchange.

Geodesic flow?

What has geodesic flow to do with a mechanical system?

The motion on the constant energy surfaces can be equivalent to a geodesic flow in negative curvature.

Example

(from youtube)

Triple Linkage Video

[Hunt, Mackay, Nonlinearity (2003)]

Initial condition

Let $v_x = \|p_x\|^{-1} p_x$, $\boldsymbol{e}_x = \frac{1}{2} p_x^2$ and use the coordinates $(q_x, v_x, \boldsymbol{e}_x)$.

With random initial conditions

$$\mathbb{E}(f) = \int f(q, v, \bar{\boldsymbol{e}}) h(q, v)\, dq\, dv$$

for given $h \in \mathcal{C}^1$ and $\bar{\boldsymbol{e}}_x > 0$.

Time scales

Let $\overline{\mathcal{E}}_{\varepsilon,x}(t) = \boldsymbol{e}_x(\varepsilon^{-1}t)$, then, by averaging, $\overline{\mathcal{E}}_{\varepsilon,x} \Rightarrow \overline{\mathcal{E}}_x$ such that

$$\frac{d}{dt}\overline{\mathcal{E}}_x = -\int \sqrt{\overline{\mathcal{E}}_x} v_x \sum_{\|x-y\|\leq 1} \nabla V(q_x, q_y) h_{\overline{\mathcal{E}}}(q,v) dq dv = 0.$$

since $h_{\bar{\boldsymbol{e}}}(q,v) = h_{\bar{\boldsymbol{e}}}(q,-v)$; $h_{\bar{\boldsymbol{e}}}$ invariant density at $\varepsilon = 0$ (Liouville).

Trivial averaged dynamics: on the ε^{-1} scale energy is conserved.

For arbitrary, but fixed, T define the random variables

$\mathcal{E}_{\varepsilon,x}(t) = \boldsymbol{e}_x(\varepsilon^{-2}t) \in \mathcal{C}^0([0,T], \mathbb{R}_+)$

Homogenization

Theorem (Dolgopyat, L.; C.M.P. (2011))

$\{\mathcal{E}_{\varepsilon,x}\}$ converges in law to $\{\mathcal{E}_x\}$ satisfying the mesoscopic SDE

$$d\mathcal{E}_x = \sum_{|x-y|=1} \boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y)dt + \sum_{|x-y|=1} \boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y)dB_{x,y}$$

$$\mathcal{E}_x(0) = \bar{\boldsymbol{e}}_x$$

where $\boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y) = -\boldsymbol{b}(\mathcal{E}_y, \mathcal{E}_x)$, $\boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y) = \boldsymbol{a}(\mathcal{E}_y, \mathcal{E}_x)$ and $B_{x,y} = -B_{y,x}$ are independent standard Brownian motions.

The limit PDE

The invariant measures has density $\boldsymbol{h}_\beta = \prod_{x\in\Lambda} \mathcal{E}_x^{\frac{\nu}{2}-1} e^{-\beta\mathcal{E}_x}$.

The SDE corresponds to a parabolic PDE with generator

$$\mathcal{L} = \frac{1}{2\boldsymbol{h}_0} \sum_{|x-y|=1} (\partial_{\mathcal{E}_x} - \partial_{\mathcal{E}_y}) \boldsymbol{h}_0 \boldsymbol{b}^2 (\partial_{\mathcal{E}_x} - \partial_{\mathcal{E}_y}).$$

That is, $\mathcal{L}^* = \mathcal{L}$ and

$$\partial_t \boldsymbol{u} = \mathcal{L}\boldsymbol{u}.$$

Hydrodynamic Limit

Let $\Lambda_L = [-L, L]^\nu \subset \mathbb{Z}^\nu$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^\nu, \mathbb{R})$ and define

$$\tilde{\mathcal{E}}_L(t, \varphi) = L^{-\nu} \sum_{x \in \Lambda_L} \mathcal{E}_x(L^2 t)\varphi(L^{-1}x)$$

Goal: prove that $\tilde{\mathcal{E}}_L(t, \varphi)$ converges weakly to $\int u(t,x)\varphi(x)$ where

$$\partial_t u = \operatorname{div}(\kappa \nabla u),$$

for some diffusion coefficient $\kappa \in \mathcal{C}^1(\mathbb{R}^\nu, GL(\nu, \mathbb{R}))$.

$\varepsilon \to 0$, what the hell....

If $\varepsilon > 0$ small, but fixed, then the previous results do not apply

I am exchanging the limits !

Thus to progress we need to:

Understand the statistical properties of the flows generated by $\mathcal{H}_\varepsilon$.

Is it possible or is it syfy?

Simplifying life

The simplest (non trivial) case:

1. Consider the case of discrete, rather than continuous time.
2. Consider as few degrees of freedom as possible: one for the fast variable, one for the slow.
3. Realise the “complex dynamics” of the fast degree of freedom via the simplest possible example of “chaotic” map.
4. Assume that the full dynamics takes place in a compact space.

A toy model

$$F_\varepsilon(x,z) = (f(x,z), z + \varepsilon\omega(x,z)),$$

dynamics: $(x_n, z_n) = F_\varepsilon^n(x_0, z_0)$ with initial conditions

$$\mathbb{E}(g(x_0,z_0)) = \int_{\mathbb{T}^1} \rho(x) g(x,\bar{z}) dx \qquad \bar{z} \in \mathbb{T}^1.$$

1. $\partial_x f(x,z) \geq \lambda > 1$ (expanding map)
2. F_0 has z as a *conserved quantity*
3. $\rho \in \mathcal{C}^2(\mathbb{T}^1, \mathbb{R}_+)$

Chaos

The above hypotheses imply that

1. for each $z \in \mathbb{T}^1$, $f(\cdot, z)$ has a unique physical measure μ_z absolutely continuous w.r.t. Lebesgue with density $h(\cdot, z)$
2. $h \in \mathcal{C}^3(\mathbb{T}^2, \mathbb{R}_+)$
3. for each $z \in \mathbb{T}$, $f(\cdot, z)$ enjoys exponential decay of correlations

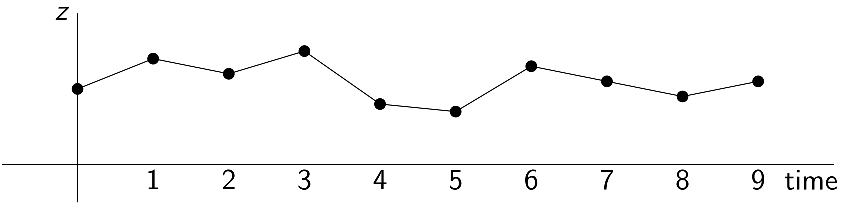
Discrete ⟶ continuous

Note that we have $z_n = z_0 + \varepsilon \sum_{k=0}^{n-1} \omega(x_k, z_k)$, thus

$$z_n - z_m = \mathcal{O}(\varepsilon(n-m)).$$

Introduce the macroscopic time $t = \varepsilon n$ and the continuous paths

$$z_\varepsilon(t) = z_{\lfloor \varepsilon^{-1} t \rfloor} + (\varepsilon^{-1} t - \lfloor \varepsilon^{-1} t \rfloor)(z_{\lfloor \varepsilon^{-1} t \rfloor + 1} - z_{\lfloor \varepsilon^{-1} t \rfloor}), \quad t \in [0, T].$$



Averaging

Since the $\{z_\varepsilon\}$ are uniformly Lipschitz they have convergent subsequences in $\mathcal{C}^0([0,T],\mathbb{R})$. The accumulation points $\bar z$ satisfy

$$\dot{\bar z} = \bar\omega(\bar z)$$

$$\bar z(0) = \bar z$$

$$\bar\omega(z) = \int_{\mathbb{T}^1} \omega(x,z)h(x,z)dx,$$

hence they are unique and the limit exists.

Anosov (1960) and Bogolyubov-Mitropolskii (1961).

Fluctuations (linear noise)

Let $\zeta_\varepsilon(t) = \frac{z_\varepsilon(t)-\bar{z}(t)}{\sqrt{\varepsilon}}$ (fluctuations around the average).

The decay of correlations implies

$$\mathbb{E}([\zeta_\varepsilon(t) - \zeta_\varepsilon(s)]^4) \le C|t-s|^2.$$

Hence, by Kolmogorov criteria, the sequence is tight.

Fluctuations (linear noise)

The accumulation points ζ of ζ_ε satisfy

$$d\zeta = \bar{\omega}'(\bar{z}(t))\zeta(t)dt + \sigma(\bar{z}(t))dB$$

$$\zeta(0) = 0$$

where $\sigma > 0$ is given by an appropriate Green-Kubo formula.

Results of this type have been first obtained by Dolgopyat (2004) but see recent generalisations by Melbourne (2013).

Noise (non-linear)

We have seen that $z_\varepsilon \sim \bar{z} + \sqrt{\varepsilon}\zeta$. But $\bar{z} + \sqrt{\varepsilon}\zeta \sim \tilde{z}_\varepsilon$, where $\tilde{z}_\varepsilon$ satisfies

$$d\tilde{z}_\varepsilon = \bar{\omega}(\tilde{z}_\varepsilon)dt + \sqrt{\varepsilon}\sigma(\tilde{z}_\varepsilon)dB,$$

an SDE investigated by Wentzell–Freidlin and Kifer (70's-80's).

Morally, this is the equivalent of the energy diffusion equation.

What $\sim$ really means? For how long does it hold?

Noise (quantitative)

Theorem ($^{\text{de Simoi, L.; Invent. (to appear)}}_{\text{de Simoi, L., Poquet, Volk; J.S.P. (2017)}}$)

There exists $\alpha \in (0,1)$ *and a coupling* $\mathbb{P}_c$ *: for all* $\varepsilon > 0$ *and* $t \leq \varepsilon^{-\alpha}$

$$\mathbb{P}_c(|z_\varepsilon(t) - \tilde{z}_\varepsilon(t)| \geq \varepsilon)| \leq C\varepsilon^{\alpha}.$$

Up to the scale ε and time $\varepsilon^{-\alpha}$, stochastic = deterministic.

Can we obtain informations for longer times?

Statistical properties

Example: consider $\bar{\omega}$ with only two non-degenerate zeroes.

Theorem (De Simoi, L.; Invent. 2016)

If the central Lyapunov exponent is negative, then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the system has a unique physical measure μ. *Moreover, for each $f, g \in C^1(\mathbb{T}^2, \mathbb{R})$*

$$|\mu(f \circ F_\varepsilon^n \cdot g) - \mu(f)\mu(g)| \le C_\# e^{-C_\# \frac{\varepsilon}{\ln \varepsilon^{-1}} n}.$$

Back to the future

The above example provides a proof of concept:

it may be possible to obtain similar results for Hamiltonian systems

relevant to the problem of Energy Transport