Planar Ising model at criticality:

State-of-the-art and perspectives

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Planar Ising model at criticality: outline

• **Combinatorics**
  Definition, phase transition
  Dimers and fermionic observables
  Spin correlations and fermions on double-covers
  Kadanoff–Ceva’s disorders and propagation equation
  Diagonal correlations and orthogonal polynomials

• **Conformal invariance at criticality**
  S-holomorphic functions and Smirnov's s-harmonicity
  Spin correlations: convergence to tau-functions
  More fields and CFT on the lattice
  Convergence of interfaces and loop ensembles
  Tightness of interfaces and ‘strong’ RSW

• **Beyond regular lattices: s-embeddings [2017+]**

• **Perspectives and open questions**
Planar Ising model: definition [Lenz, 1920]

- **Lenz-Ising model** on a planar graph $G^*$ (dual to $G$) is a random assignment of $+/−$ spins to vertices of $G^*$ (=faces of $G$) according to

$$\mathbb{P} \left[ \text{conf. } \sigma \in \{±1\}^{V(G^*)} \right] \propto \exp \left[ \beta \sum_{e=\langle uv \rangle} J_{uv} \sigma_u \sigma_v \right] = Z^{-1} \cdot \prod_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv},$$

where $J_{uv} > 0$ are interaction constants preassigned to edges $\langle uv \rangle$, $\beta = 1/kT$, and $x_{uv} = \exp[-2\beta J_{uv}]$.

- **Remark:** w/o magnetic field $⇒$ ‘free fermion’.

- **Example:** homogeneous model ($x_{uv} = x$) on $\mathbb{Z}^2$.
  - Ising’25: no phase transition in 1D $\rightsquigarrow$ doubts;
  - Peierls’36: existence of the phase transition in 2(+)-D;
  - Kramers-Wannier’41: $x_{\text{self-dual}} = \sqrt{2} − 1$;
  - Onsager’44: sharp phase transition at $x_{\text{crit}} = x_{\text{self-dual}}$.

Ensemble of domain walls between ‘$+$’ and ‘$−$’ spins.

- ‘$+$’ boundary conditions $⇒$ collection of loops.
Planar Ising model: **phase transition** [Kramers–Wannier’41: $x_{\text{crit}} = \sqrt{2} - 1$ on $\mathbb{Z}^2$]

- **Spin-spin correlations:**
  - e.g., two spins at distance $2n \to \infty$ along a diagonal.
  - $x < x_{\text{crit}}$: does not vanish;
  - $x = x_{\text{crit}}$: **power-law** decay;
  - $x > x_{\text{crit}}$: exponential decay.

**Theorem** [“diagonal correlations”, Kaufman–Onsager’49, Yang’52, McCoy–Wu’66+]:

(i) For $x = \tan \frac{1}{2} \theta < x_{\text{crit}}$, one has $\lim_{n \to \infty} \mathbb{E}^{\mathbf{C}^0}[\sigma_0 \sigma_{2n}] = (1 - \tan^4 \theta)^{1/4} > 0$.

(ii) At criticality, $\mathbb{E}^{\mathbf{C}^0}_{x=x_{\text{crit}}}[\sigma_0 \sigma_{2n}] = \left(\frac{2}{\pi}\right)^n \prod_{k=1}^{n-1} \left(1 - \frac{1}{4k^2}\right)^{k-n} \sim C_\sigma^2 \cdot (2n)^{-\frac{1}{4}}$.

**Remark:** Many highly nontrivial results on the **spin correlations in the infinite volume** are known. Reference: B.M.McCoy – T.T.Wu “The two-dimensional Ising model”.
Planar Ising model: phase transition \cite{Kramers-Wannier'41}: $x_{\text{crit}} = \sqrt{2} - 1$ on $\mathbb{Z}^2$

- **Spin-spin correlations:**
  - e.g., two spins at distance $2n \to \infty$ along a diagonal.
  - $x < x_{\text{crit}}$: does not vanish;
  - $x = x_{\text{crit}}$: **power-law** decay;
  - $x > x_{\text{crit}}$: exponential decay.

- **Domain walls structure:**
  - $x < x_{\text{crit}}$: “straight”;
  - $x = x_{\text{crit}}$: SLE(3), CLE(3);
  - $x > x_{\text{crit}}$: SLE(6), CLE(6).
    - [this is not proved]
Combinatorics: planar Ising model via dimers ('60s) and fermionic observables

Fisher’s graph $G^F$: vertices are corners and oriented edges of $G$.

- **Kasteleyn’s theory**: $F = \overline{F} = -F^\top$, $\mathcal{Z} \cong \text{Pf}[F]

- **Fermions**: $\langle \phi_c \phi_d \rangle := F^{-1}(c, d) = -\langle \phi_d \phi_c \rangle$

  Pfaffian (or Grassmann variables) formalism:

  $$\langle \phi_{c_1} \cdots \phi_{c_{2k}} \rangle = \text{Pf}[\langle \phi_{c_p} \phi_{c_q} \rangle]_{p,q=1}^{2k}$$
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$G^C$: vertices = corners of $G$.

Kasteleyn’s terminal graph $G^K$, vertices = oriented edges of $G$.
Combinatorics: planar Ising model via dimers ('60s) and fermionic observables

Fisher’s graph $G^F$: vertices are corners and oriented edges of $G$.

There are other combinatorial correspondences of the same kind:

$$\mathcal{Z} \cong \text{Pf}[F]$$
$$\cong \text{Pf}[K]$$
$$\cong \text{Pf}[C]$$

Two other useful techniques:

- **Kac–Ward matrix** is equivalent to $K$;
- **Smirnov’s fermionic observables (2000s)** are combinatorial expansions of $\text{Pf}[F_{V(G^F)\backslash\{c,d\}}]$.

Kasteleyn’s terminal graph $G^K$, vertices = oriented edges of $G$.


“Revisiting the combinatorics of the 2D Ising model”
**Combinatorics:** spin correlations and fermions on double-covers

Fisher’s graph $G^F$: vertices are corners and oriented edges of $G$.

**Observation:**

$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] = \frac{\text{Pf}[F_{[u_1,\ldots,u_n]}]}{\text{Pf}[F]}$$

One changes $x_e \rightarrow -x_e$ along $\gamma_{[u_1,u_2]}$ to compute $\mathbb{E}[\sigma_{u_1} \sigma_{u_2}]$. 
Combinatorics: spin correlations and fermions on double-covers

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Observation: 
$$E[\sigma_{u_1} \cdots \sigma_{u_n}] = \frac{\text{Pf}[F_{[u_1,..,u_n]}]}{\text{Pf}[F]}$$

One changes $x_e \mapsto -x_e$ along $\gamma[u_1,u_2]$ to compute $E[\sigma_{u_1} \sigma_{u_2}]$.

Corollary: Let $w_1 \sim u_1$. The ratio 
$$\frac{E[\sigma_{w_1} \sigma_{u_2} \cdots \sigma_{u_n}]}{E[\sigma_{u_1} \sigma_{u_2} \cdots \sigma_{u_n}]}$$
can be expressed via $F_{[u_1,..,u_n]}^{-1}$.

Remark: Instead of fixing cuts one can view $F_{[u_1,..,u_n]}^{-1}(c^b, d) = -F_{[u_1,..,u_n]}^{-1}(c^d, d)$ as a spinor on the double-cover $G^F_{[u_1,..,u_n]}$ of the graph $G^F$ ramified over faces $u_1, .., u_n$. 
Combinatorics: Kadanoff–Ceva ('71) disorders and propagation equation

• Given (an even number of) vertices $v_1, \ldots, v_m$, consider the Ising model on (the faces of) the double-cover $G[v_1, \ldots, v_m]$ ramified over $v_1, \ldots, v_m$ with the spin-flip symmetry constraint $\sigma_u^\flat = -\sigma_u^\sharp$ provided that $u^\flat, u^\sharp$ lie over the same face $u$ of $G$.

• Define $\langle \mu_{v_1} \cdots \mu_{v_m} \sigma_{u_1} \cdots \sigma_{u_n} \rangle$

  $$:= \mathbb{E}[v_1, \ldots, v_m][\sigma_{u_1} \cdots \sigma_{u_n}] \cdot Z[v_1, \ldots, v_m] / Z.$$

[!] By definition, this (formal) correlator changes the sign when one of $u_k$ goes around of one of $v_s$.

[c] Clément Hongler (EPFL)
Combinatorics: Kadanoff–Ceva (’71) disorders and propagation equation

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- Define \( \langle \mu_{v_1} \cdots \mu_{v_m} \sigma_{u_1} \cdots \sigma_{u_n} \rangle \):
  \[
  \langle \mu_{v_1} \cdots \mu_{v_m} \sigma_{u_1} \cdots \sigma_{u_n} \rangle := \mathbb{E}[v_1, \ldots, v_m][\sigma_{u_1} \cdots \sigma_{u_n}] \cdot \mathcal{Z}[v_1, \ldots, v_m] / \mathcal{Z}.
  \]

- For a corner \( c \) of \( G \), define \( \chi_c := \mu_{v(c)} \sigma_{u(c)} \).

- **Proposition:** If all vertices \( v(c_k) \) are distinct, then
  \[ \pm \langle \chi_{c_1} \cdots \chi_{c_{2k}} \rangle = \pm \langle \phi_{c_1} \cdots \phi_{c_{2k}} \rangle. \]

**Proof:** expand both sides combinatorially on \( G \).

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Combinatorics: Kadanoff–Ceva('71) disorders and propagation equation

Parameterization:

$$x_e = \tan \frac{1}{2} \theta_e$$

- **Propagation equation:** Let $X(c) := \langle \chi_c O[\mu, \sigma] \rangle$. Then $X(c_{00}) = X(c_{01}) \cos \theta_e + X(c_{10}) \sin \theta_e$.

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[two disorders inserted]

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  [Perk’80, Dotsenko–Dotsenko’83, . . ., Mercat’01]

- **Bosonization:** To obtain a combinatorial representation of the model via *dimers on* \( G^D \) one should start with *two Ising configurations* [e.g., see Dubédat’11, Boutillier–de Tilière’14]

\[ G^D : \text{bipartite (Wu–Lin'75).} \]

**Fact:** \( D^{-1} = C^{-1} + \text{local} \).

\[ G^C : \text{vertices} = \text{corners of} \ G. \]
Infinite-volume limit on $\mathbb{Z}^2$: diagonal correlations and orthogonal polynomials

- The propagation equation implies the (massive) harmonicity of spinors on each type of the corners.
- Fourier transform allows to construct such a spinor explicitly.
- Its values on $\mathbb{R}$ must be coefficients of an orthogonal polynomial

**Theorem** [“diagonal correlations”, Kaufman–Onsager’49, Yang’52, McCoy–Wu’66+]:

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(ii) At criticality, $E_{\mathbb{C}^\diamond}^{x=x_{\text{crit}}} [\sigma_0 \sigma_{2n}] = \left( \frac{2}{\pi} \right)^n \prod_{k=1}^{n-1} \left( 1 - \frac{1}{4k^2} \right)^{k-n} \sim C_{\sigma}^2 \cdot (2n)^{-\frac{1}{4}}$.

**Remark:** Originally considered as a very involved derivation, nowadays it can be done in two pages (see arXiv:1605:09035), based on the strong Szegö theorem for simple real weights on $\mathbb{T}$. 
Conformal invariance at $x_{\text{crit}}$: $s$-holomorphicity

Assume that each $(v_0 u_0 v_1 u_1)$ is drawn as a rhombus with an angle $2\theta_{v_0 v_1}$ and $x_e = \tan \frac{1}{2} \theta_e$

- **Propagation equation:** Let $X(c) := \langle \chi_c O[\mu, \sigma] \rangle$. Then $X(c_{00}) = X(c_{01}) \cos \theta_e + X(c_{10}) \sin \theta_e$.

**Remark:** In particular, this setup includes
- square ($x_{\text{crit}} = \sqrt{2} - 1 = \tan \frac{\pi}{8}$),
- honeycomb ($x_{\text{crit}} = 1/\sqrt{3} = \tan \frac{\pi}{6}$),
- triangular ($x_{\text{crit}} = 2 - \sqrt{3} = \tan \frac{\pi}{12}$) and
- rectangular ($2x_h/(1-x_h^2) \cdot 2x_v/(1-x_v^2) = 1$) grids.

**Critical $Z$-invariant model**
[Baxter’86] on isoradial graphs:
[... Boutillier–deTilière–Raschel’16]
Conformal invariance at $x_{\text{crit}}$: s-holomorphicity

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- S-holomorphicity: Let $F(c) := \eta_c \delta^{-1/2} X(c)$, where $\eta_c := e^{i \pi/4} \exp[-i \frac{1}{2} \arg(v(c) - u(c))]$.

Then $F(c) = \Pr[F(z); \eta_c] = \frac{1}{2} \left[ F(z) + \eta_c^2 F(z) \right]$ for some $F(z) \in \mathbb{C}$ and all corners $c \sim z$.

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- **A priori regularity theory for s-holomorphic functions** [Ch.–Smirnov’09] is based on the following miraculous fact:

- **Smirnov’s s-harmonicity:** Let $F$ be s-holomorphic. Then $\Delta \cdot H_F \geq 0, \quad \Delta^\circ H_F \leq 0$, where $H_F = \int \text{Im}[F(z)^2dz]$. The function $H_F$ is defined by $H_F(v) - H_F(u) := (X(c))^2$ and can/should be viewed as.
Conformal invariance at $x_{\text{crit}}$: spin correlations \cite{12, w/ C. Hongler & K. Izyurov}

**Theorem:** Let $\Omega \subset \mathbb{C}$ be a (bounded) simply connected domain and $\Omega_\delta \to \Omega$ as $\delta \to 0$. Then

$$
\delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_1} \ldots \sigma_{u_n}] \mathop{\to}_{\delta \to 0} C^n_\sigma \cdot \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+ \Omega,
$$

where $\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+ = \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle^+ \Omega' \cdot \prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}}$ for conformal mappings $\varphi : \Omega \to \Omega'$ and

$$
\left[ \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+ \right]^2 = \prod_{1 \leq s \leq n} (2 \text{Im } u_s)^{-\frac{1}{4}} \cdot \sum_{\beta \in \{\pm 1\}^n} \prod_{s < m} \left| \frac{u_s - u_m}{u_s - \overline{u_m}} \right|^{\frac{\beta_s \beta_m}{2}}.
$$

**Techniques:** Analysis of the kernel $D^{-1}_{[u_1, \ldots, u_n]}$ viewed as the s-holomorphic solution to a discrete Riemann-type boundary value problem. Applying Smirnov’s trick, boundary conditions $\text{Im}[F(\zeta) \tau(\zeta)^{1/2}] = 0$ become $\int^\zeta \text{Im}[F(z)^2dz] = H_F(\zeta) = 0, \ z \in \partial \Omega$. 
Conformal invariance at $x_{\text{crit}}$: spin correlations ['12, w/ C. Hongler & K. Izyurov]

As $\delta \to 0$, one gets the isomonodromic $\tau$-function

$$\det D_{[\Omega; u_1, \ldots, u_n]} : \quad \text{where } D_{[\Omega; u_1, \ldots, u_n]} f := \partial \bar{f}$$

is an anti-Hermitian operator acting in (originally) the real Hilbert space of spinors $f : \Omega[u_1, \ldots, u_n] \to \mathbb{C}$ satisfying Riemann-type b.c. $\bar{f} = \tau f$ on $\partial \Omega$.

[Kyoto school (Jimbo, Miwa, Sato, Ueno)'70s; . . . ; Palmer'07 “Planar Ising correlations”; Dubédat’11]

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\[
\left[ \langle \sigma u_1 \ldots \sigma u_n \rangle_{\mathbb{H}} \right]^{+} = \prod_{1 \leq s \leq n} (2 \text{Im } u_s)^{-\frac{1}{4}} \cdot \sum_{\beta \in \{\pm 1\}^n} \prod_{s < m} \frac{\beta_s \beta_m}{u_s - \overline{u_m}} \cdot \prod_{s \leq m} \frac{u_s - u_m}{u_s - \overline{u_m}}.
\]

• Remark: Passing to the complex Hilbert space one gets the (massless) Dirac operator

\[
\left( \begin{array}{cc} 0 & \partial \\ -\partial & 0 \end{array} \right) \left( \begin{array}{c} f \\ \tilde{f} \end{array} \right) = \left( \begin{array}{c} \partial \overline{\tilde{f}} \\ \overline{\partial f} \end{array} \right)
\]

with b.c. $\tilde{f} = \tau f$. For $\Omega = \mathbb{H}$ this operator boils down to

\[
f \mapsto \overline{\partial f} \text{ on } \mathbb{C}_{[u_1, \ldots, u_n, \overline{u_1}, \ldots, \overline{u_n}]}.
\]

• Convergence of random distributions: Basing on the convergence of multi-point spin correlations, one can study the convergence of random fields $(\delta^{-\frac{1}{8}} \sigma_u)_{u \in \Omega_\delta}$ to a (non-Gaussian!) random Schwartz distribution on $\Omega$ [Camia–Garban–Newman ’13, Furlan–Mourrat ’16] (see also [Caravenna–Sun–Zygouras ’15] for disorder relevance results).
Conformal invariance at $x_{\text{crit}}$: more fields and CFT on the lattice

From the CFT perspective, the 2D critical Ising model is

- **FFF (= Fermionic Free Field):** $\mathcal{Z} = \text{Pf}[D]$.
- Minimal model with central charge $c = \frac{1}{2}$ and three primary fields $1, \sigma, \varepsilon$ with scaling exponents $0, \frac{1}{8}, 1$.

**Convergence results:**

- **Fermions:** [Smirnov'06 ($\mathbb{Z}^2$), Ch.–Smirnov'09 (isoradial)];
- **Energy densities:** $\varepsilon := \sqrt{2} \cdot \sigma_e - \sigma_e^\ast - 1 = \frac{i}{2} \psi_e \psi_{e}^\ast$
  [Hongler–Smirnov'10, Hongler'10];
- **Spins:** [Ch.–Hongler–Izyurov'12];

**Mixed correlations:** [Ch.–Hongler–Izyurov, ’16–’18]

spins ($\sigma$), disorders ($\mu$), fermions ($\psi, \psi^\ast$), energy densities ($\varepsilon$) in multiply connected domains $\Omega$, with **mixed fixed/free boundary conditions**. The limits of correlations are defined via solutions to appropriate Riemann-type boundary value problems in $\Omega$. 
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- **Convergence results:**
  - Fermions: [Smirnov’06 ($\mathbb{Z}^2$), Ch.–Smirnov’09 (isoradial)];
  - Energy densities: $\varepsilon := \sqrt{2} \cdot \sigma_e - \sigma_e^+ - 1 = \frac{i}{2} \psi_e \psi_e^*$ [Hongler–Smirnov’10, Hongler’10];
  - Spins: [Ch.–Hongler–Izyurov’12];
- **Mixed correlations:** [Ch.–Hongler–Izyurov, ’16–’18]

spins ($\sigma$), disorders ($\mu$), fermions ($\psi, \psi^*$), energy densities ($\varepsilon$) in multiply connected domains $\Omega$, with mixed fixed/free boundary conditions. The limits of correlations are defined via solutions to appropriate Riemann-type boundary value problems in $\Omega$.

- **And more** [Hongler–Kytölä–Viklund’17, ...]:
  E.g., one can define an action of the **Virasoro algebra** on **local lattice fields** via the Sugawara construction applied to lattice fermions.
Conformal invariance at $x_{\text{crit}}$: interfaces and loop ensembles

- Dobrushin b.c., weak topology:
  [Smirnov'06], [Ch.–Smirnov'09]
- Dipolar SLE(3) (+/free/− b.c.):
  [Hongler–Kytölä’11], [Izyurov’14]
- Strong topology (tightness of curves):
  [Kemppainen–Smirnov’12]
- Brief summary up to that date:

- Spin-Ising boundary arc ensemble for free b.c.: [Benoist–Duminil-Copin–Hongler’14]
- Convergence of the full spin-Ising loop ensemble to CLE(3): [Benoist–Hongler’16]
- Exploration of FK boundary loops: [Kemppainen–Smirnov’15], see also [Garban–Wu’18]
- Convergence of the FK loop ensemble to CLE(16/3): [Kemppainen–Smirnov’16]
- “CLE percolations” [Miller–Sheffield–Werner’16]: FK-Ising $\rightsquigarrow$ CLE(16/3) $\rightsquigarrow$ CLE(3)
Conformal invariance at $x_{\text{crit}}$: interfaces and loop ensembles

- Dobrushin b.c., weak topology:
  [Smirnov’06], [Ch.–Smirnov’09]

- Dipolar SLE(3) (+/free/− b.c.):
  [Hongler–Kytölä’11], [Izyurov’14]

- Strong topology (tightness of curves):
  [Kemppainen–Smirnov’12]

- Brief summary up to that date:

\textbf{Theorem [Smirnov’06]:}

\begin{itemize}
  \item Ising interfaces $\rightarrow$ SLE(3)
  \item FK-Ising ones $\rightarrow$ SLE(16/3)
\end{itemize}

\textbf{Fortuin–Kasteleyn (random cluster) expansion of the spin-Ising model [Edwards–Sokol coupling]:}

spins $\sim$ FK: $p_e := 1 - x_e$ percolation on spin clusters;
FK $\sim$ spins: toss a fair coin for each of the FK clusters.
Conformal invariance at $x_{\text{crit}}$: CLE(3) = ? [Sheffield–Werner, arXiv:1006.2374]

• **Question:** What could be a good candidate for the *scaling limit of the outermost domain walls* surrounding ‘—’ clusters in $\Omega_\delta$ (with ‘+’ b.c.)?

• **Intuition:** This random loop ensemble should
  (a) be *conformally invariant*;
  (b) satisfy the *domain Markov property*: given the loops intersecting $D_1 \setminus D_2$, the remaining ones form the same CLEs in the complement.

• **Theorem:** Provided that its loops do not touch each other, a CLE must have the following law for some intensity $c \in (0, 1]$:
  (i) sample a (countable) set of *Brownian loops* using the natural conformally-friendly *Poisson process* of intensity $c$;
  (ii) fill the *outermost clusters*.

• **Nesting:** Iterate the construction inside all the *first-level loops*. 
Conformal invariance at $x_{\text{crit}}$: convergence of loop ensembles

- **Subtlety in the passage from SLEs to CLEs:**
  To prove the convergence to a CLE, one uses an iterative *exploration procedure* (e.g., [B–H’16] alternate between exploring boundary arc ensembles for free b.c. and FK-Ising clusters touching the boundary).

  To ensure that discrete and continuous exploration processes do not deviate from each other (e.g., to control relevant *stopping times*), one needs uniform *crossing estimates* in rough domains [‘strong’ RSW]

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- Spin-Ising boundary arc ensemble for free b.c.: [Benoist–Duminil-Copin–Hongler’14]
- Convergence of the full spin-Ising loop ensemble to **CLE(3):** [Benoist–Hongler’16]
- Exploration of FK boundary loops: [Kemppainen–Smirnov’15], see also [Garban–Wu’18]
- Convergence of the FK loop ensemble to **CLE(16/3):** [Kemppainen–Smirnov’16]
- “CLE percolations” [Miller–Sheffield–Werner’16]: FK-Ising $\rightsquigarrow$ CLE(16/3) $\rightsquigarrow$ CLE(3)
Conformal invariance at $x_{\text{crit}}$: tightness of interfaces

- **Crossing estimates (RSW):** due to the FKG inequality it is enough to prove that
  \[
  P \left[ \begin{array}{c}
  \end{array} \right] \geq \eta(k) > 0
  \]
  for rectangles of a given aspect ratio $k > \sqrt{3} + 1$, uniformly over all scales.

  \[ \Downarrow \]
  [Aizenman–Burchard’99]
  \[ \Downarrow \]
  [Kemppainen–Smirnov’12]

  Arm exponents $\Delta_n \geq \varepsilon n \Rightarrow$ tightness of curves and of the corresponding Loewner driving forces $\xi^\delta_t$: 
  \[
  \mathbb{E} [ \exp (\varepsilon |\xi_t^\delta|/\sqrt{t}) ] \leq C.
  \]
Conformal invariance at $x_{\text{crit}}$: tightness of interfaces and ‘strong’ RSW

- **Crossing estimates (RSW):** due to the FKG inequality it is enough to prove that

$$\mathbb{P}\left[\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{crossing}}
\end{array}\right] \geq \eta(k) > 0$$

for rectangles of a given aspect ratio $k > \sqrt{3}+1$, uniformly over all scales.

$\downarrow$ [Aizenman–Burchard’99]
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Arm exponents $\Delta_n \geq \varepsilon n \Rightarrow$ tightness of curves and of the corresponding Loewner driving forces $\xi_t^\delta$: $\mathbb{E}[\exp(\varepsilon|\xi_t^\delta|/\sqrt{t})] \leq C$.

**Theorem:** [Ch.–Duminil-Copin–Hongler’13]
Uniformly w.r.t. $(\Omega_\delta; a, b, c, d)$ and b.c.,

$$\mathbb{P}^{\text{FK}}[(ab) \leftrightarrow (cd)] \geq \eta(L_{\Omega;\delta}(ab),(cd)) > 0,$$

where $L_{\Omega;\delta}(ab),(cd)$ is the discrete extremal length (= effective resistance) of the quad.

- **Remark:** Such a uniform lower bound is not straightforward even for the random walk partition functions [‘toolbox’: arXiv:1212.6205].
Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

- **Question:** generalize convergence results from the very particular isoradial case to (as) general (as possible) weighted graphs.

- **A model question:** (reversible) random walks in a periodic (or in your favorite) environment.

- **Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.
Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

**Question:** generalize convergence results from the very particular isoradial case to (as) general (as possible) weighted graphs.

**A model question:** (reversible) random walks in a periodic (or in your favorite) environment.

**But ... how should we draw a planar graph?**
- Invariance under the star-triangle transform;
- Compatibility with the isoradial setup.

**Random walks:** Tutte’s barycentric embeddings.

[!] For periodic graphs, we also need to fix the conformal modulus of the fundamental domain.

**Planar Ising model:** s-embeddings.

**Criticality:** $x(\mathcal{E}_0) = x(\mathcal{E}_1)$
[Cimasoni–Duminil-Copin’12]

\[
1 + x_3 x_4 = x_3 + x_4 + x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_4 + x_1 x_2 x_3 x_4
\]

**Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.

horizontal: $x_1, x_2; \quad$ vertical: $x_3, x_4.$
Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

Assume that each \((v_0 u_0 v_1 u_1)\) is a rhombus with an angle \(2\theta_{v_0 v_1}\) and \(x_e = \tan \frac{1}{2} \theta_e\).

- **Propagation equation:**
  \[
  X(c_{00}) = X(c_{01}) \cos \theta_e + X(c_{10}) \sin \theta_e.
  \]

- **S-holomorphism:** Let \(F(c) := \eta_c \delta^{-1/2} X(c)\), where \(\eta_c := e^{i \pi \frac{3}{4}} \exp[-\frac{i}{2} \arg(v(c) - u(c))]\).

[!] In the isoradial setup, \(X(c) := (v(c) - u(c))^{1/2}\) satisfies the propagation equation.

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  [Cimasoni–Duminil-Copin’12]
  \[
  1 + x_3 x_4 = x_3 + x_4 + x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_4 + x_1 x_2 x_3 x_4
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- **Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.
How to draw graphs: (periodic) s-embeddings

At criticality, the propagation equation admits two periodic solutions.

- Propagation equation:
  \[ X(c_{00}) = X(c_{01}) \cos \theta_e + X(c_{10}) \sin \theta_e. \]

- Definition: Given a (periodic) complex-valued solution \( \lambda' \) to the PE, we define the s-embedding \( S_{\lambda'} \) of the graph by \( S_{\lambda'}(v) - S_{\lambda'}(u) := (\lambda(c))^2 \).

- The function \( L_{\lambda'}(v) - L_{\lambda'}(u) := |\lambda(c)|^2 \) is also well-defined \( \Rightarrow \) tangential quads.

- Criticality: \( x(\mathcal{E}_0) = x(\mathcal{E}_1) \) [Cimasoni–Duminil-Copin’12]
  \[ 1 + x_3 x_4 = x_3 + x_4 + x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_4 + x_1 x_2 x_3 x_4 \]

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Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

At criticality, the propagation equation admits two periodic solutions.

- Propagation equation:
  \[ X(c_{00}) = X(c_{01}) \cos \theta_e + X(c_{10}) \sin \theta_e. \]

- Definition: Given a (periodic) complex-valued solution \( \mathcal{X} \) to the PE, we define the s-embedding \( S_{\mathcal{X}} \) of the graph by
  \[ S_{\mathcal{X}}(v) - S_{\mathcal{X}}(u) := (\mathcal{X}(c))^2. \]

- S-holomorphism:
  \[ e^{i \frac{\pi}{4}} X(c) / \mathcal{X}(c) = \text{Pr}[F(z);\eta_c] \]
  for all real-valued spinors \( X \) satisfying the PE.

\[ S_{\mathcal{X}}(v) - S_{\mathcal{X}}(u) := (\mathcal{X}(c))^2 \]
\[ L_{\mathcal{X}}(v) - L_{\mathcal{X}}(u) := |\mathcal{X}(c)|^2 \]

- Lemma: \( \exists! \mathcal{X} : L_{\mathcal{X}} \rightarrow \text{periodic}. \)

- Theorem [Ch., 2018]: The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.
Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

- **Key ingredients:**
  - A priori Lipshitzness of projections Pr\[F(z); \alpha]\;
  - Control of discrete contour integrals of \( F \) via \( L_X \);
  - Positivity lemma: \( \Delta_S H_F \geq 0 \) for some \( \Delta_S = \Delta_S^\top \) ([!] \( \Delta_S \) is sign-indefinite \( \Leftrightarrow \) no interpretation via RWs);
  - A priori regularity of \( H_F \) is nevertheless doable;
  - Coarse-graining for \( H_F \): harmonicity in the limit;
  - Boundedness of \( F \) near “straight” boundaries \( \Rightarrow \) convergence for (special) “straight” rectangles;
  - \( \Rightarrow RSW \Rightarrow \) convergence for arbitrary shapes \( \Omega \).

- **S-holomorphicty:** \( e^{i \frac{\pi}{4}} X(c)/\mathcal{X}(c) = \Pr[F(z);\eta_c] \)
  for all real-valued spinors \( X \) satisfying the PE.

\[
S\mathcal{X}(v) - S\mathcal{X}(u) := (\mathcal{X}(c))^2
\]
\[
L\mathcal{X}(v) - L\mathcal{X}(u) := |\mathcal{X}(c)|^2
\]

- **Lemma:** \( \exists! \mathcal{X} : L\mathcal{X} - \text{periodic.} \)

- **Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to \( \text{SLE}(16/3) \) holds for all periodic weighted graphs.
Some perspectives and open questions

**periodic setup:** other observables, ‘strong’ RSW, loop ensembles, spin correlations;

**your favorite object in your favorite setup:** invariance principle for the limit;

**Ising model on random planar maps:** can one attack not only SLEs/CLEs but also LQG in this way?

- **Topological correlators** in the planar Ising model and CLE(3): is it possible to understand the convergence of ‘topological correlators’ for loop ensembles directly via a kind of $\tau$-functions?

- **Supercritical regime, renormalization:** convergence to CLE(6) for $x > x_{\text{crit}}$. 
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THANK YOU FOR YOUR ATTENTION!