CONFORMAL AND DLR MEASURES ON MARKOV SUBSHIFTS WITH INFINITELY MANY STATES

Ruy Exel

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This talk is based on ongoing joint work with Rodrigo Bissacot, Rodrigo Frausino and Thiago Raszeja from the University of São Paulo.

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Markov's shift is the map $\sigma : \Sigma_A \to \Sigma_A$, given by

$$\sigma(x_1 x_2 x_3 ...) = x_2 x_3 x_4 ...$$

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The dual Markov operator acts on the space of Borel measures on Σ_A, as follows: given a Borel measure μ on Σ_A, we define $L_\beta^*(\mu)$ to be the measure on Σ_A such that

$$\int_{\Sigma_A} f \, dL_\beta^*(\mu) = \int_{\Sigma_A} L_\beta(f) \, d\mu, \quad \forall\, f \in C(\Sigma_A)$$

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That is, μ should be an eigen-measure of L_β^*. When $\lambda = 1$, these are called conformal measures.

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Our goal is to be able to study infinite state Markov spaces without having to abandon the tools of topology.

Cuntz-Krieger algebras.

Given a separable, infinite dimensional Hilbert space $\mathcal{H}$, and given any positive integer n, we may split $\mathcal{H}$ as

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If no row of A is identically zero, these are also separable, infinite dimensional spaces. We may then choose, for every i, an isometric isomorphism

$$S_i : \bigoplus_{j \,:\, A_{i,j}=1} \mathcal{H}_j \to \mathcal{H}_i$$

For example:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

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Extending each S_i to $\mathcal{H}$ by setting it to be zero on the orthogonal complement of its original domain, we get a family $\{S_1, S_2, \ldots, S_n\}$ of bounded operators on $\mathcal{H}$, satisfying

$$S_i S_i^* S_i = S_i, \qquad \sum_{j=1}^{n} S_j S_j^* = I, \qquad S_i^* S_i = \sum_{j=1}^{n} A_{i,j} S_j S_j^*.$$

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Definition. The Cuntz-Krieger algebra, denoted $\mathcal{O}_A$, is a C*-algebra generated by a family of operators $\{S_1, S_2, \ldots, S_n\}$ satisfying the above relations in an "universal way".

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One may prove that the elements of the form $S_\alpha S_\alpha^*$ are pairwise commuting projections and hence they generate a commutative C*-sub-algebra

$$\mathcal{D}_A \subseteq \mathcal{O}_A.$$

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Under the natural isomorphism $\mathcal{D}_A \simeq C(\Sigma_A)$, each $S_\alpha S_\alpha^*$ is identified with the characteristic function of the cylinder

$$\Big\{(x_1, x_2, x_3, \ldots) \in \Sigma_A : x_i = \alpha_i, \text{ for } i = 1, \ldots, |\alpha|\Big\}.$$

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Thus we may view $C(\Sigma_A) = \mathcal{D}_A \subseteq \mathcal{O}_A$, and if we let

$$S = n^{-1/2} \sum_{i=1}^{n} S_i,$$

one may prove that

$$Sf = (f \circ \sigma)S, \quad \forall f \in C(\Sigma_A),$$

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In other words, $\mathcal{O}_A$ encodes the Markov shift in its algebraic structure!

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In the paper "Cuntz-Krieger algebras for infinite matrices" [Crelle 1999], joint with Marcelo Laca, we were able to figure out the general "non row-finite" case.

Given any matrix $A = \{A_{i,j}\}_{i,j \in I}$, for each $i \in I$, let S_i be the bounded operator on the Hilbert space $\ell^2(\Sigma_A)$ given on the orthonormal basis $\{\delta_\omega\}_{\omega \in \Sigma_A}$ by

$$S_i(\delta_\omega) = \begin{cases} \delta_{i\omega}, & \text{if } i\omega \in \Sigma_A, \\ 0, & \text{otherwise.} \end{cases}$$

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In other words, $\mathcal{D}_A = C(X_A)$, for some compact space X_A, which could be thought of as an alternative for the badly behaved Markov space Σ_A.

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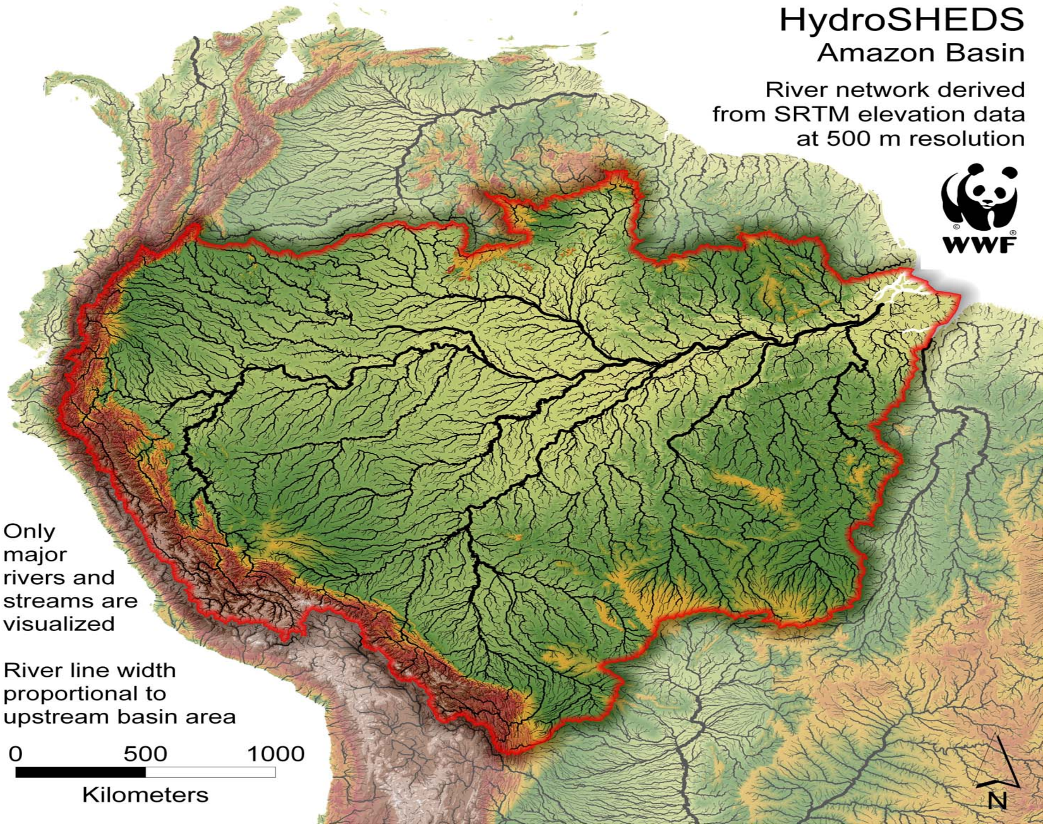
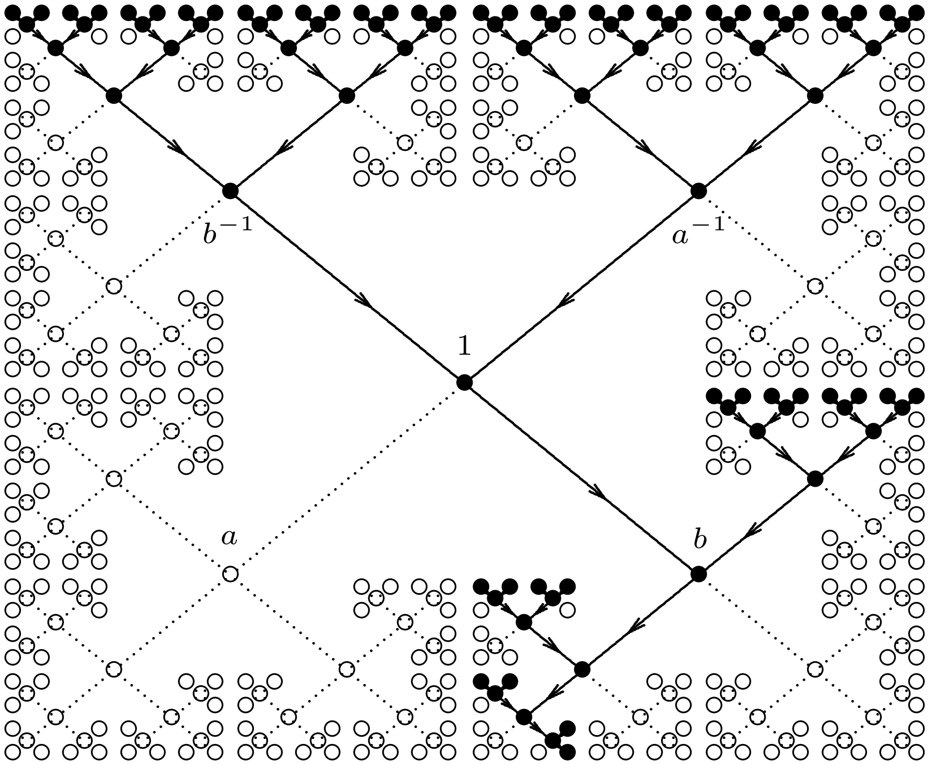

HydroSHEDS
Amazon Basin
River network derived
from SRTM elevation data
at 500 m resolution
WWF
Only
major
rivers and
streams are
visualized
River line width
proportional to
upstream basin area
0
500
1000
Kilometers
N



A picture of $\xi_\omega \subseteq \mathbb{F}$

Here is the Cayley graph of $\mathbb{F}$. The generators point to $\swarrow$ and $\searrow$

The given word ω is the main "river", starting at the group unit. It then recieves a lot of "tributaries", namely all possible rivers which merge into the main river forming an admissble word.

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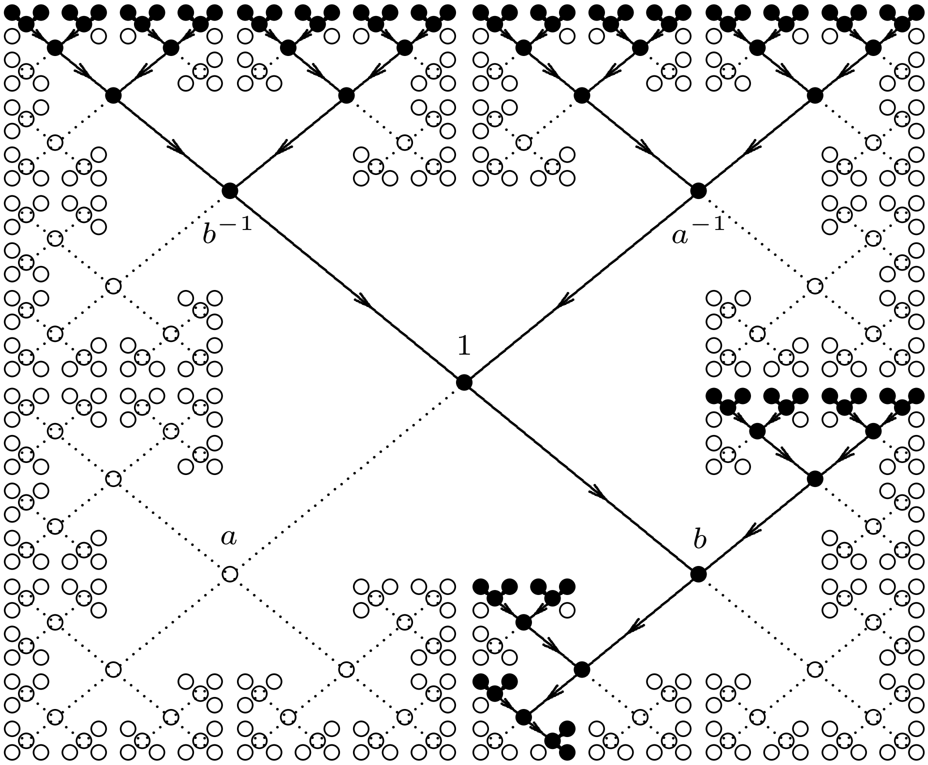
The new elements in the closure also look like river basins but the main river may dry up, that is, it may be a finite word! In particular it may dry up at its very source, namely at 1.



b^{-1}
a^{-1}
1
a
b

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In "Cuntz-like algebras" [Timișoara, 1998] Jean Renault realized that this local homeo encodes all of the relevant information. In particular Renault showed that $\mathcal{O}_A$ is the C*-algebra for the generalized Deaconu-Renault groupoid associated to σ.

Since the publication of the first paper on this subject in the late 90's, Marcelo Laca and I expected that this would be of interest to people studying Markov chains with infinitely many states.

Until recently we have not had much success, but I have recently been in São Paulo working with a very active group of Mathematical Physicists led by Rodrigo Bissacot, and what follows is a bit of what we have discovered.

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Given a continuos potential $h : U \to \mathbb{R}$, we may define a 1-cocycle on $\mathcal{G}_\sigma$ by

$$c(x, n-m, y) = \sum_{i=0}^{n-1} h(\sigma^i(x)) - \sum_{j=0}^{m-1} h(\sigma^j(y)).$$

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A 1-cocycle induces a flow, i.e., a strongly continuous one parameter group of automorphisms on the groupoid C*-algebra $C^*(\mathcal{G}_\sigma)$, as follows:

$$\alpha_t(f)|_\gamma = \mathrm{e}^{itc(\gamma)} f(\gamma), \quad \forall\, f \in C_c(\mathcal{G}_\sigma), \quad \forall\, \gamma \in \mathcal{G}_\sigma.$$

It is a problem of fundamental importance to find the probability measures μ on X, such that the associated state

$$\varphi_\mu(f) = \int_X f(x)\, d\mu(x), \quad \forall\, f \in C_c(\mathcal{G}_\sigma),$$

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By this we mean the following: define measures ν_r and ν_s on $\mathcal{G}_\sigma$ by

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(iv) μ satisfies the Denker-Urbanski condition $\dfrac{d\mu \odot \sigma}{d\mu} = \mathrm{e}^{\beta h}$, where $\mu \odot \sigma$ is the unique measure on U such that

$$(\mu \odot \sigma)(E) = \mu(\sigma(E)),$$

for every Borel set E, such that σ is injective on E.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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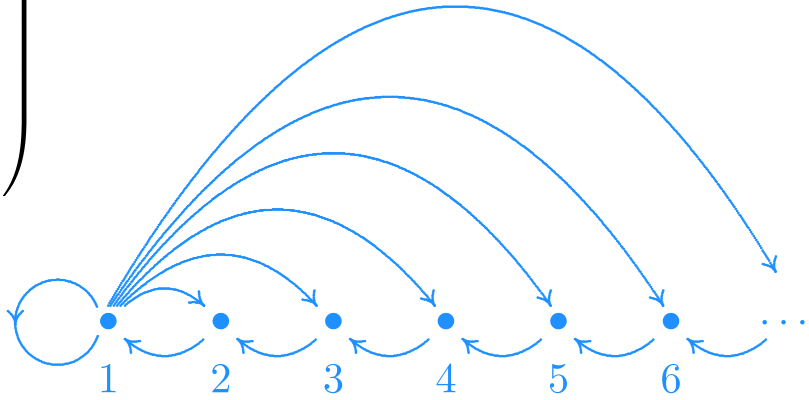

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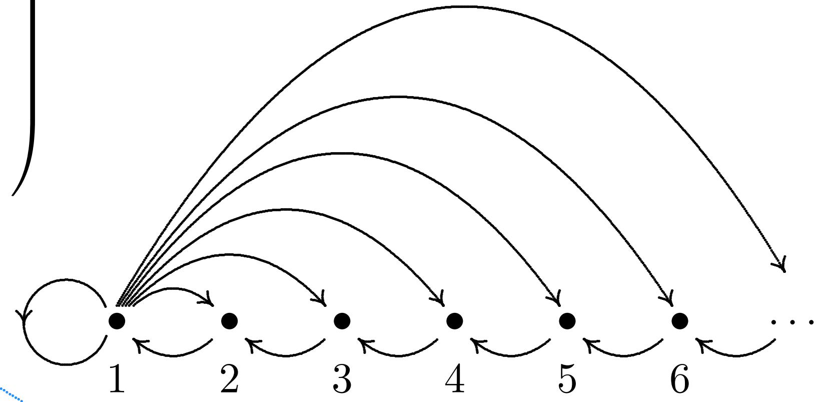

Notice that this matrix is not row-finite!

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Moreover $X_A = \Sigma_A \cup Y_A$, where

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$$c^0 = e^\beta c_1, \quad \text{and} \quad c_{\sigma(\omega)} = e^\beta c_\omega,$$

for every admissible ω ending in 1, where $\sigma(\omega)$ is the shift, deleting the first letter of the finite word ω.

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MORAL: for ∞-state-Markov shifts, there may be conformal measures which cannot be seen within Σ_A. To see them, one must pass from Σ_A to X_A.

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(1) we must work with non compact X,

We would now like to look at the sub-groupoid of $\mathcal{G}_\sigma$ formed by the elements of the form $(x, 0, y)$. Notice that such an element lies in $\mathcal{G}_\sigma$, iff there exists some n, such that $x, y \in \text{dom}(\sigma^n)$, and $\sigma^n(x) = \sigma^n(y)$. This highlights an equivalence relation R_n on $\text{dom}(\sigma^n)$ according to which

$$(x, y) \in R_n \iff \sigma^n(x) = \sigma^n(y).$$

An equivalence relation R on a topological space X is said to be proper if the quotient space is Hausdorff and the quotient map is a local homeomorphism.

In that case R is an étale groupoid with the topology inherited from the product topology on $X \times X$.

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(2) each R_n lives on a different set, namely $\text{dom}(\sigma^n)$.

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Also every U_m is invariant under every R_n in the sense that

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Given U_n and R_n as above, we have that $R := \cup_i R_i$ is an equivalence relation on X which becomes an étale groupoid with the inductive limit topology.

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The crucial point is then to understand quasi-invariant measures for a single proper equivalence relation on a locally compact space.

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Fortunately we are saved by Lebesgue's theory of integration which does not worry too much about functions taking values in $[0, \infty]$.

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(v) there exits a positive measure ν on U, such that

$$\int_U \zeta\,d\nu = 1, \quad \text{and} \qquad \int_U f\,d\mu = \int_U E_\beta(f)\,d\nu, \quad \forall\, f \in C_c(U).$$

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Moreover, if the conditions above are satisfied, then

$$\mu\{x \in U : \zeta(x) = \infty\} = 0.$$

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Then F_β is a conditional expectation and we see that (iv) becomes the usual DLR (Dobrushin–Lanford–Ruelle) condition

$$\mu = F_\beta^*(\mu)$$

Returning to the Deaconu-Renault groupoid $\mathcal{G}_\sigma$ for a given $\sigma : U \subseteq X \to X$, the subgroupoid formed by the triples $(x, 0, y)$ turns out to be the groupoid for the generalized AP equivalence relation where

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If c is the cocycle on $\mathcal{G}_\sigma$ defined by a potential $h : X \to \mathbb{R}$, as above, then the restriction of c to R coincides with the cocycle given by the family of potentials

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It therefore follows that the flow defined by c on $C^*(\mathcal{G}_\sigma)$ leaves $C^*(R)$ invariant and its restriction to $C^*(R)$ coincides with the flow defined by the k_n.

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KMS states on $C^*(\mathcal{G}_\sigma)$ therefore restrict to KMS states on $C^*(R)$, and quasi-invariant measures for $\mathcal{G}_\sigma$ are obviously quasi-invariant for R.

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In other words, every conformal measure on X satisfies the generalized DLR condition (Theorem B.iv), namely: for every n, and for every f in $C_c(U_n)$,

$$\int_{U_n} f\, d\mu = \int_{U_n} \sum_{(x,y)\in R_n} \frac{e^{\beta h_n(x)}}{\zeta_n(x)} f(x)\, d\mu(y),$$

where

$$h_n(x) = \sum_{i=0}^{n-1} h(\sigma^i(x)), \quad \text{and} \quad \zeta_n(y) = \sum_{(x,y)\in R_n} e^{\beta h_n(x)}.$$

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