Rigidity for von Neumann algebras

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von Neumann algebras

A **von Neumann algebra** is an algebra of bounded operators on a Hilbert space $\mathcal{H}$ which is closed under adjoint and in the weak operator topology:

$$T_i \to T \ \text{ w.o.t. } \quad \text{if} \quad \langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle, \quad \text{for all} \quad \xi, \eta \in \mathcal{H}.$$

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Examples

- $\mathbb{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$.
- $\mathrm{L}^\infty(X) \subset \mathbb{B}(\mathrm{L}^2(X))$, where (X, μ) is a measure space.
- The commutant $\mathcal{A}'$ of any set $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ closed under adjoint.
- **von Neumann's bicommutant theorem**:
  If $M \subset \mathbb{B}(\mathcal{H})$ is a von Neumann algebra and $\mathrm{Id} \in M$, then $M = (M')'$.

General constructions (Murray & von Neumann, 1936-43)

- Γ countable group $\rightsquigarrow$ group von Neumann algebra $L(\Gamma)$.

  Generated by the left regular representation $\{\lambda_g\}_{g\in\Gamma}$:
  λ_g is the unitary operator on $\ell^2(\Gamma)$ given by $\lambda_g(\delta_h) = \delta_{gh}$.

  $L(\Gamma)$ is the weak operator closure of $\mathbb{C}\Gamma = \text{span}\{\lambda_g\}_{g\in\Gamma}$.

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- $\Gamma \curvearrowright (X, \mu)$ measure preserving action on a probability space (X, μ)
  $\rightsquigarrow$ group measure space von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$.

  Generated by a copy of $\mathrm{L}^\infty(X)$ and a copy of Γ subject
  to the relations $\lambda_g F \lambda_g^* = F \circ g^{-1}$ for $g \in \Gamma$ and $F \in \mathrm{L}^\infty(X)$.

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Fact

These algebras admit a **trace**: linear functional satisfying $\tau(ST) = \tau(TS)$.

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Proposition

- $\mathrm{L}(\Gamma)$ is a II_1 factor if and only if Γ has infinite conjugacy classes (icc).
- $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a II_1 factor if $\Gamma \curvearrowright (X, \mu)$ is free and ergodic.

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Murray and von Neumann (1936-43):

1. $\exists!$ approximately finite dimensional II_1 factor $R = \overline{\bigotimes_{n\in\mathbb{N}} \mathbb{M}_2(\mathbb{C})}^{\text{w.o.t.}}$
2. $\mathrm{L}(\mathbb{F}_2) \not\cong R$, where $\mathbb{F}_2$ is the free group on two generators.

The amenable case

Definition

A group Γ is **amenable** if its left regular representation has almost invariant vectors: there exist unit vectors $\xi_n \in \ell^2(\Gamma)$ satisfying $\|\lambda_g(\xi_n) - \xi_n\|_2 \to 0$, for all $g \in \Gamma$.

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Examples. abelian and solvable groups.
Remark. To see that $\mathbb{Z}$ is amenable, take $\xi_n = \frac{1}{\sqrt{n}} \mathbf{1}_{\{1,2,\dots,n\}} \in \ell^2(\mathbb{Z})$.

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Theorem (Connes, 1975)

- *$L(\Gamma)$ is isomorphic to R, for every icc amenable group Γ.*
- *$L^\infty(X) \rtimes \Gamma$ is isomorphic to R, for every free ergodic action of an infinite amenable group Γ.*

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Striking lack of rigidity: any algebraic property of an amenable group (e.g. being torsion free) is lost when passing to von Neumann algebras.

The non-amenable case

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A group Γ has **Kazhdan's property (T)** if any unitary representation of Γ with almost invariant vectors has non-zero invariant vectors.
Examples. Lattices in higher rank simple Lie groups, e.g. $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$.

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Popa's deformation/rigidity theory (2001-)

General idea: Study II_1 factors M that have both a

- **deformation property**, e.g. M has a large group of automorphisms.
- **rigidity property**, e.g. M contains $L(\Gamma)$ for a property (T) group Γ.

Combine these properties to understand the structure of M.

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$\rightsquigarrow$ Rigidity results: when various aspects of groups Γ and actions $\Gamma \curvearrowright (X, \mu)$ are remembered by their von Neumann algebras.

Rigidity for group measure space II_1 factors

Definition

Two free ergodic actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are

1. **conjugate** if there exist isomorphisms $\alpha : (X, \mu) \to (Y, \nu)$ and $\delta : \Gamma \to \Lambda$ such that $\alpha(g \cdot x) = \delta(g) \cdot \alpha(x)$.

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3. **W*-equivalent** if $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$.

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Remark. conjugacy $\Longrightarrow$ orbit equivalence $\xRightarrow{\textbf{Singer (1955)}}$ W^*-equivalence.

Rigidity: whenever the implications can be reversed for a class of actions.

Popa's strong rigidity theorem

Theorem (Popa, 2004)

Let Γ be a property (T) group and $\Gamma \curvearrowright (X, \mu)$ a free ergodic action.
Let Λ be an icc group and $\Lambda \curvearrowright (Y, \nu) = (Y_0, \nu_0)^\Lambda$ a Bernoulli action.

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An action $\Gamma \curvearrowright (X,\mu)$ is called **W*-superrigid** if any action $\Lambda \curvearrowright (Y,\nu)$ such that $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ is conjugate to $\Gamma \curvearrowright (X,\mu)$.

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- **Peterson (2009)**: existence of virtually W*-superrigid actions.
- **Popa and Vaes (2009)**: first concrete families of W*-superrigid actions: Bernoulli actions of many amalgamated free product groups.

W*-superrigidity for Bernoulli actions

Theorem (I, 2010)

If Γ is any icc property (T) group, then the Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ is W*-superrigid.

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Therefore, for a large family of actions, the von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$ remembers the group and the action.

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Groups Γ covered by the theorem:
$PSL_n(\mathbb{Z})$ and $PSL_n(\mathbb{Z}) \times \Sigma$ (for all $n \geq 3$ and Σ icc), $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$.

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Remark. The free group $\mathbb{F}_n$ has no W*-superrigid actions: any action of $\mathbb{F}_n$ is orbit equivalent to uncountably many different actions of $\mathbb{F}_n$.

Rank rigidity for free group measure space II_1 factors

Theorem (Popa and Vaes, 2011)

Let $\mathbb{F}_n \curvearrowright (X, \mu)$ and $\mathbb{F}_m \curvearrowright (Y, \nu)$ be any free ergodic actions.

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- **Gaboriau (1999):** Actions of free groups of different ranks are never orbit equivalent.

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Major open problems

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$$\Downarrow$$

Connes' conjecture is equivalent to: Any icc property (T) group Γ is **W*-superrigid**: any group Λ such that $L(\Gamma) \cong L(\Lambda)$ is isomorphic to Γ.

W*-superrigid groups

Theorem (I, Popa and Vaes, 2010)

Let G_0 be any non-amenable group and S be any infinite amenable group.
Define the wreath product $G = G_0 \wr S := (\bigoplus_{s \in S} G_0) \rtimes S$.
Consider the action $G \curvearrowright I = G/S$ by left multiplication.

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Then $\Gamma = (\bigoplus_{i \in I} \mathbb{Z}/2\mathbb{Z}) \rtimes G$ is **W*-superrigid***.*

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Theorem (I, Popa and Vaes, 2010)

Let G_0 be any non-amenable group and S be any infinite amenable group. Define the wreath product $G = G_0 \wr S := (\bigoplus_{s \in S} G_0) \rtimes S$.

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Question: What algebraic properties of groups can be recovered from their von Neumann algebras?

Prime II_1 factors

Definition. A II_1 factor M is prime if $M \neq M_1 \bar{\otimes} M_2$, $\forall$ II_1 factors M_1, M_2.
Note: If $\Gamma = \Gamma_1 \times \Gamma_2$, then $\mathrm{L}(\Gamma) = \mathrm{L}(\Gamma_1) \bar{\otimes} \mathrm{L}(\Gamma_2)$.

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If Γ is any icc hyperbolic group or any icc lattice in a simple Lie group of rank one, then $L(\Gamma)$ is prime.

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Conjecture

Show that $\mathrm{L}(\mathrm{PSL}_m(\mathbb{Z}))$ is prime, for any $m \geqslant 3$.
Show that $\mathrm{L}(\Gamma)$ is prime, for any icc irreducible lattice $\Gamma < G$ in a product of simple Lie groups $G = G_1 \times ... \times G_n$.

Prime II_1 factors from irreducible lattices

Theorem (Drimbe, Hoff and I., 2016)

Let $G = G_1 \times ... \times G_n$, with $G_1, ..., G_n$ simple Lie groups of rank one.
Let $\Gamma < G$ any icc irreducible lattice. Then $L(\Gamma)$ is a prime II_1 factor.

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Examples of irreducible lattices

- $SL_m(\mathbb{Z}) < SL_m(\mathbb{R})$, for $m \geqslant 2$.
- $SL_m(\mathbb{Z}[\sqrt{d}]) < SL_m(\mathbb{R}) \times SL_m(\mathbb{R})$, $d \geq 2$ square-free, $d \not\equiv 1 \pmod 4$.

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Corollary. $\mathrm{L}(\mathrm{PSL}_2(\mathbb{Z}[\sqrt{d}]))$ is a prime II_1 factor.

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Remark

If $L(\Gamma)$ is prime, then Γ is not a product of infinite groups.
Thus, $L(\Gamma)$ remembers the absence of a product decomposition for Γ.

Algebraic rigidity

Theorem (Chifan, de Santiago and Sinclair, 2015)

Let $\Gamma = \Gamma_1 \times ... \times \Gamma_n$, where $\Gamma_1, ..., \Gamma_n$ are icc hyperbolic groups.

If $L(\Gamma) \cong L(\Lambda)$, then $\Lambda = \Lambda_1 \times ... \times \Lambda_n$ is a product of n infinite groups.

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Theorem (Chifan and I., 2017)

For a class of amalgamated free product groups $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ we have:

If $L(\Gamma) \cong L(\Lambda)$, then $\Lambda = \Lambda_1 *_\Delta \Lambda_2$ is an amalgamated free product group. Moreover, we have $L(\Gamma_1) \cong L(\Lambda_1)$, $L(\Gamma_2) \cong L(\Lambda_2)$ and $L(\Sigma) \cong L(\Delta)$.

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