Conformal field theory, vertex operator algebras and operator algebras

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Operator algebraic approach to conformal field theory

Interactions between two approaches to chiral conformal field theory: vertex operator algebras and local conformal nets. (The latter is based on operator algebras.)

Joint work with S. Carpi, R. Longo and M. Weiner (Mem. Amer. Math. Soc., 2018).

Outline of the talk:

1. Moonshine and vertex operator algebras
2. Quantum fields and chiral conformal field theory
3. Local conformal nets
4. Representation theory
5. From vertex operator algebras to local conformal nets and back

The Monster and the j-function

The Monster group is one of the 26 sporadic finite simple groups and has the largest order, around 8×10^{53}, among them.

The following function, called j-function, has been classically studied.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

For $q = \exp(2\pi i\tau)$, $\mathrm{Im}\,\tau > 0$, this is characterized by modular invariance property, $j(\tau) = j\left(\dfrac{a\tau+b}{c\tau+d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, and starting with q^{-1}.

Vertex operator algebras (VOA)

The first part of the Moonshine conjecture, proved by Frenkel-Lepowsky-Meurman, asserts that we have some "natural" infinite dimensional graded vector space $V^\natural = \bigoplus_{n=0}^\infty V_n^\natural$ over $\mathbb{C}$ with $\dim V_n^\natural < \infty$ having some algebraic structure whose automorphism group is the Monster group and that the series $\sum_{n=0}^\infty (\dim V_n^\natural) q^{n-1}$ is the j-function minus 744.

A vertex operator algebra gives a precise axiomatization of the above "some algebraic structure". Each $u \in V$ is assumed to produce a vertex operator, $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$, where $u_n \in \mathrm{End}(V)$ and z is a formal variable. We have $u_n v \in V_{k+l-n-1}$ for $u \in V_k$, $v \in V_l$. This is the n-th product of u and v.

Examples of vertex operator algebras

Basic methods of constructions

1. Affine Kac-Moody algebras (Frenkel-Zhu)
2. Virasoro algebra (Frenkel-Zhu)
3. Even lattices (Frenkel-Lepowsky-Meurman)

New constructions from given examples

1. Tensor product construction
2. Orbifold construction (Dijkgraaf-Vafa-Verlinde-Verlinde)
3. Simple current extension (Schellekens-Yankielowicz)
4. Coset construction (Frenkel-Zhu)
5. Extension by a Frobenius algebra (Huang-Kirillov-Lepowsky) (2015)

Quantum Field Theory: (mathematical aspects/axioms)

Mathematical ingredients: Spacetime, its symmetry group, quantum fields on the spacetime.

From a mathematical viewpoint, quantum fields are certain operator-valued distributions on the spacetime. Axiomatization of such operator-valued distributions on a Hilbert space is given by the Wightman axioms.

A pairing $\langle T, f\rangle$ for a quantum field T and a test function f supported in O gives an observable in O. For a fixed O, let $A(O)$ be the von Neumann algebra generated by these observables. We have a family $\{A(O)\}$ of von Neumann algebras of bounded linear operators parameterized with regions O, and it is called a net. We work on its mathematical axiomatization.

Chiral conformal field theory

This is a quantum field theory on S^1 with the spacetime symmetry group $\mathrm{Diff}(S^1)$. It is described with a family $\{A(I)\}$ of von Neumann algebras parameterized by an interval $I \subset S^1$ subject to certain axioms. Such a family is called a local conformal net.

Axioms for a local conformal net:

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy, vacuum vector
5. Irreducibility

The locality axiom comes from the Einstein causality. Each $A(I)$ is automatically the Araki-Woods III_1 factor.

Examples of local conformal nets

Basic methods of constructions

1. Affine Kac-Moody algebras (Gabbiani-Fröhlich, Wassermann)
2. Virasoro algebra (Loke, Xu)
3. Even lattices (K-Longo, Dong-Xu)

New constructions from given examples

1. Tensor product construction
2. Orbifold construction (Xu, Moonshine by K-Longo)
3. Simple current extension (Böckenhauer-Evans)
4. Coset construction (Xu)
5. Extension by a Frobenius algebra/Q-system (Longo-Rehren, K-Longo, Xu)

Representation theory

Let $\{A(I)\}$ be a local conformal net . Each $A(I)$ acts on the same Hilbert space from the beginning. Consider a representation of these von Neumann algebras on another Hilbert space.

A classical Doplicher-Haag-Roberts theory adapted to a local conformal net shows that each representation gives a subfactor of the Jones theory, and the representation theory produces a braided tensor category.

We are often interested in a situation where we have only finitely many irreducible representations. (Rationality.) K-Longo-Müger gave an operator algebraic characterization of complete rationality of a local conformal net. (We then get a modular tensor category.)

α-induction

We recall a classical notion of induction of a representation of a group and introduce a similar construction for a local conformal net.

Let $\{A(I) \subset B(I)\}$ be an inclusion of local conformal nets. We can produce an almost representation $\alpha_\lambda^\pm$ of $\{B(I)\}$ from a representation λ of $\{A(I)\}$, using the $\pm$-braiding. ($\alpha^\pm$-induction: Longo-Rehren, Xu, Ocneanu, Böckenhauer-Evans-K)

Böckenhauer-Evans-K have shown that the matrix $(Z_{\lambda,\mu})$ defined by $Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$ is a modular invariant for a $SL(2, \mathbb{Z})$ representation arising from the braiding of the representation category.

Classification theory for small central charge values

Using modular invariants, K-Longo have obtained the following complete classification of local conformal nets with $c < 1$, where c is a numerical invariant called the central charge arising from the Virasoro algebra.

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.
(2) Their simple current extensions with index 2.
(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$.

Three of the four exceptionals in the above (3) are identified with coset constructions, but the remaining one $c = 144/145$ does not seem to be related to any other previously known constructions so far, and is given as an extension by a Frobenius algebra.

Unitarity and energy bounds

Now we construct a local conformal net from a VOA V. We first need a positive definite inner product for a Hilbert space. We have to assume to have such one, like for many natural examples. This is called unitarity.

Let V be a unitary VOA. The meaning of $Y(u, z)$ should be a Fourier expansion of an operator-valued distribution on S^1. We say that $u \in V$ satisfies energy-bounds if we have positive integers s, k and a constant $M > 0$ such that we have

$$\|u_n v\| \leq M(|n| + 1)^s \|(L_0 + 1)^k v\|,$$

for all $v \in V$ and $n \in \mathbb{Z}$. If every $u \in V$ satisfies energy-bounds, we say V is energy-bounded.

Strong locality

For every $u \in V$, we define the operator $Y_0(u, f)$ by $Y_0(u, f)v = \sum_{n\in\mathbb{Z}} \hat{f}_n u_n v$ for $v \in V$, where f is a C^∞ function supported in $I \subset S^1$, $\hat{f}_n$ is its Fourier coefficient and the sum is convergent.

Let $Y(u, f)$ be the closure of $Y_0(u, f)$ on the completion of V. The (possibly unbounded) operators $Y(u, f)$, where $u \in V$ and $\mathrm{supp}\ f \subset I$, generate a von Neumann algebra $A(I)$. The family $\{A(I)\}$ satisfies all the axioms of a local conformal net except for locality. (Conformal covariance is nontrivial.)
If we also have locality, we say the original VOA has strong locality. The name "strong" comes from strong commutativity in functional analysis.

When do we have strong locality?

If a unitary VOA V has a set of nice generators in V_1 and V_2, we have strong locality. (Carpi-K-Longo-Weiner)

This sufficient condition applies to a VOA arising from an affine Lie algebra or the Virasoro algebra. Strong locality passes to a tensor product and a sub VOA (hence a coset construction and an orbifold construction, in particular). These together show that many known examples of VOAs are strongly local.

No example of a VOA which is not strongly local is known so far. If such an example exists, it would not correspond to a chiral conformal field theory in the physical sense.

Going back to a VOA

Suppose we have constructed a local conformal net from a strongly local VOA. We now would like to recover the original VOA.

Based on an idea of Fredenhagen-Jörss and with help of the Tomita-Takesaki theory, we can recover a smeared vertex operator $Y(u, f)$ for $u \in V$ and a test function $f \in C^\infty(S^1)$ supported in I, using one of the Virasoro generators, L_{-1}.

Then the vector space V is first recovered as an algebraic direct sum of the eigenspaces of the Virasoro generator L_0. In this way, we can also recover $u_n \in \mathrm{End}(V)$, which gives the entire structure of a VOA. (Carpi-K-Longo-Weiner)

Realization problem of a local conformal net

The representation category is an invariant similar to the higher relative commutants of subfactors. (cf. Popa's work.) It also has some formal similarity to K-theory of C^*-algebras and the flow of weights of type III factors.

The assignment map of the representation category cannot be injective, since there are many local conformal nets having the trivial representation category. But can it be surjective? That is, is a given (unitary) modular tensor category realized as the representation category of a local conformal net?

We believe the answer is positive. This would imply there would be a huge variety of new exotic chiral conformal field theories through exotic subfactors.