Structure of nuclear C*-algebras: From quasidiagonality to classification, and back again

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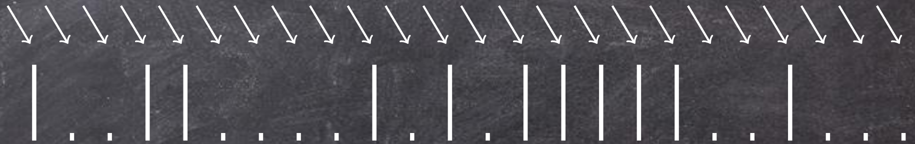

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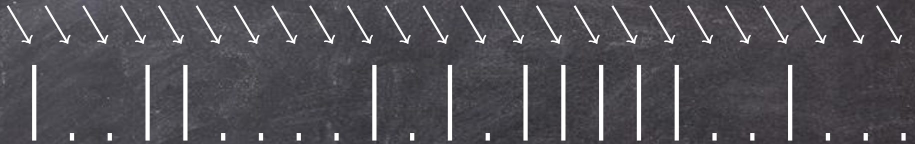

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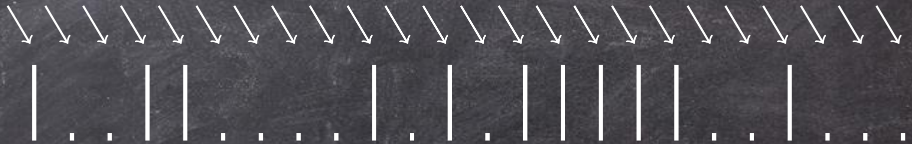

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Shift again:

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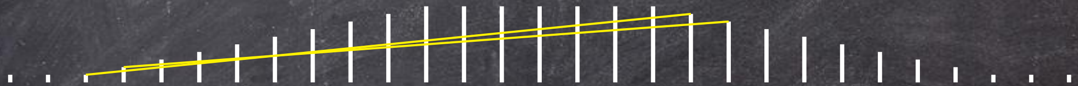

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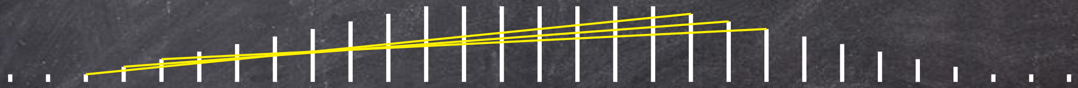

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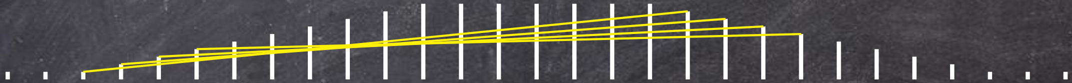

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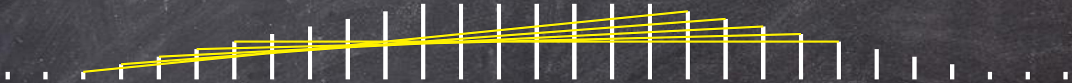

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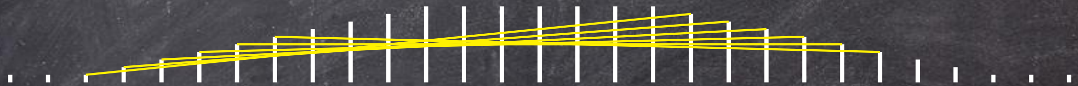

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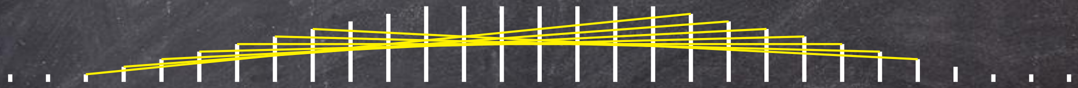

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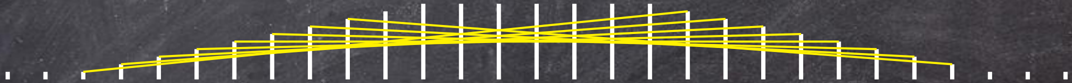

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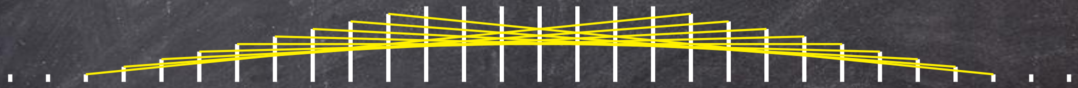

This gives an almost shift invariant finite rank projection which looks like:

Now let G be a countable discrete group and consider the left regular representation

$$\boldsymbol{\lambda} : G \longrightarrow \mathcal{B}(\ell^2(G)).$$

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OBSERVATION [Rosenberg, 1986]
Suppose $(p_n)_{\mathbb{N}} \subset \mathcal{B}(\ell^2(G))$ is an increasing sequence of finite rank projections with $p_n \nearrow_{\text{s.o.t.}} 1$ such that for every $g \in G$

$$\|p_n \boldsymbol{\lambda}(g) - \boldsymbol{\lambda}(g)\, p_n\| \stackrel{n\to\infty}{\longrightarrow} 0.$$

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PROOF
Take a free ultrafilter ω on $\mathbb{N}$ and check that

$$f \longmapsto \lim_{\omega} \operatorname{tr}(p_n\, f\, p_n)$$

is an invariant mean on $\ell^\infty(G)$. ▮

CONJECTURE [Rosenberg, late 1980s]
If G is amenable, there is an increasing sequence $(p_n)_{\mathbb{N}} \subset \mathcal{B}(\ell^2(G))$ of finite rank projections with $p_n \nearrow_{\text{s.o.t.}} 1$ such that for every $g \in G$

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The crucial condition involves norms of commutators, so the conjecture says that for an amenable discrete group its *reduced group $C^\star$-algebra*

$$\mathrm{C}^\star_{\mathrm{r}}(G) := \overline{\mathrm{span}(\boldsymbol{\lambda}(G))}^{\|\cdot\|} \subset \mathcal{B}(\ell^2(G))$$

is quasidiagonal.

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For G elementary amenable, $C^\star_r(G)$ is quasidiagonal.

The proof uses the classification theory of *simple nuclear* $C^\star$-algebras (although $C^\star_r(G)$ usually is not simple at all).

DEFINITION

A C*-algebra is a Banach *-algebra A satisfying

$$\|a^\star a\| = \|a\|^2$$

for all $a \in A$.

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$C^\star_r(G)$ is nuclear iff G is amenable.

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We will look at *finite nuclear dimension* and at $\mathcal{Z}$*-stability*.

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A $C^\star$-algebra A has nuclear dimension at most d, $\dim_{\mathrm{nuc}} A \leq d$, if there is a net

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- for each λ, $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(d)}$ and for each $k \in \{0, \ldots, d\}$, $\varphi_\lambda|_{F_\lambda^{(k)}}$ is contractive and has *order zero*, i.e., preserves orthogonality.

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[Castillejos–Evington–Tikuisis–White–W, 2018] for arbitrary trace spaces.

THEOREM [Kirchberg, 1994; Elliott–Gong–Lin–Niu, 2015]

$\Big\{ A$ separable, simple, unital, finite nuclear dimension, UCT, all traces quasidiagonal$\Big\}$

is classified by the Elliott invariant

$$\Big(\mathrm{K}_0(A),\ \mathrm{K}_0(A)_+,\ [1_A]_0,\ \mathrm{K}_1(A),\ \mathrm{T}(A),\ \mathrm{r}_A : \mathrm{T}(A) \to \mathrm{S}(\mathrm{K}_0(A))\Big).$$

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‘A satisfies the conditions of the Universal Coefficient Theorem of Rosenberg–Schochet’, i.e., for any σ-unital B there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_\star(A), \mathrm{K}_\star(B)) \longrightarrow \mathrm{KK}_\star(A, B) \longrightarrow \mathrm{Hom}(\mathrm{K}_\star(A), \mathrm{K}_\star(B)) \longrightarrow 0$$

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QUESTION
Does every separable nuclear C*-algebra satisfy the UCT?

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A is quasidiagonal iff there is an embedding

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Is every separable, nuclear, stably finite C*-algebra quasidiagonal?

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THEOREM [many hands]

$$\left\{ A \otimes \mathcal{Z} \mid A \text{ separable, simple, unital, nuclear, with UCT} \right\}$$

is classified by the Elliott invariant.

For G amenable, $C^\star_r(G)$ satisfies the UCT; it also has a faithful tracial state.

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COROLLARY
A discrete group G is amenable iff $C^\star_r(G)$ is quasidiagonal.

Let us briefly go back to the quasidiagonality theorem.

THEOREM [Tikuisis–White–W, 2015]
Let A be a separable, nuclear C*-algebra satisfying the UCT.
Then every faithful trace τ_A on A is quasidiagonal.

For the proof, first find *-homomorphisms

$$\acute{\alpha} : C_0((0,1]) \otimes A \longrightarrow \mathbf{\Pi}_{\mathbb{N}}\,\mathcal{Q} \quad \text{and} \quad \grave{\beta} : C_0([0,1)) \otimes A \longrightarrow \mathbf{\Pi}_{\mathbb{N}}\,\mathcal{Q}$$

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$$\acute{\alpha} : \mathrm{C}_0((0,1]) \otimes A \longrightarrow \mathbf{\Pi}_{\mathbb{N}}\, \mathcal{Q} \qquad \text{and} \qquad \grave{\beta} : \mathrm{C}_0([0,1)) \otimes A \longrightarrow \mathbf{\Pi}_{\mathbb{N}}\, \mathcal{Q}$$

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This is subtle; it uses Connes' classification of injective II_1-factors.

Restriction to small open subintervals and rescaling yields maps of the form

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$$\mathrm{KK}(\alpha) = \mathrm{KK}(\beta) = 0,$$

by 'shortening' the homotopies.

Restriction to small open subintervals and rescaling yields maps of the form

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Now we apply a *stable uniqueness theorem* [Dadarlat–Eilers] to arrive at the result.