Completeness of exponentials

Beurling-Malliavin and Type Problems

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Completeness problems for exponentials

Let μ be a finite positive Borel measure on $\mathbb{R}$, $\Lambda \subset C$. When is $\mathcal{E}_\Lambda = \{\exp(i\lambda t) | \ \lambda \in \Lambda\}$ complete in $L^p(\mu)$?

The Beurling-Malliavin (BM) Problem

$\Lambda = \{\lambda_n\}$ is a sequence, μ is Lebesgue measure on an interval, $p = 2$:
When is $\mathcal{E}_{\{\lambda_n\}}$ complete in $L^2(0, a)$? ($\exists ?\ f \in PW_{a/2}, \widehat{f} = 0$ on Λ)

The Type Problem

$\Lambda = [0, a]$ is an interval, μ is any finite positive measure on $\mathbb{R}$, $p = 2$:
When is $\mathcal{E}_{[0,a]}$ complete in $L^2(\mu)$? ($\exists ?\ f \in L^2(\mu)$, $\widehat{f\mu} = 0$ on $[0, a]$)

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BM Problem: Main examples

Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of real points, $\mathcal{E}_\Lambda = \{e^{i\lambda_n t}\}$. Find

$$R(\Lambda) = \sup\{a \mid \mathcal{E}_\Lambda \text{ is complete in } L^2(0,a)\}.$$

The general problem easily reduces to the case $\Lambda \subset \mathbb{R}$. Main examples:

$$R(\mathbb{Z}) = 2\pi, \quad R(2\mathbb{Z}) = \pi, \quad R(\mathbb{N}) = 2\pi.$$

Theorem (Paley and Wiener, 1934)

$$R(\Lambda) \geqslant 2\pi \limsup_{x\to\infty} \frac{\#(\Lambda \cap (0,x))}{x}.$$

Converse? Couneterexample by Levinson, 1940.

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Question

Does there exist a sequence Λ of (two-sided) upper density 0 with $R(\Lambda) = \infty$?

Kahane (1959): Yes.

Modified Question

Does there exist $\Lambda \subset \mathbb{Z}$ of upper density 0 with $R(\Lambda) = 2\pi$?

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BM Exterior Density and BM Theorem

If $\{I_n\}$ is a sequence of disjoint intervals on $\mathbb{R}$, we call it short if

$$\sum \frac{|I_n|^2}{1+\mathrm{dist}^2(0, I_n)} < \infty$$

and long otherwise. For $\Lambda \subset \mathbb{R}$ define

Exterior BM density: $D^*(\Lambda) = \sup\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \geqslant d|I_n|\}, \forall n\}$,

(Note that $\frac{1}{d}\mathbb{Z}$ has about $d|I|$ points on I.)

Theorem (Beurling and Malliavin, 1961)

Let Λ be a discrete real sequence. Then

$$R(\Lambda) = 2\pi D^*(\Lambda).$$

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Extensions and Applications of BM Theory: Other completeness problems

θ is inner in $\mathbb{C}_+$ if it is analytic, bounded and $|\theta| = 1$ a.e. on $\mathbb{R}$. Let $H^2 = H^2(\mathbb{C}_+)$ be the Hardy space in $\mathbb{C}_+$. For every inner θ one defines the model space $K_\theta = H^2 \ominus \theta H^2$. Such spaces are the main object of Nagy-Foias functional model theory.

Reproducing kernels of K_θ

If $\lambda \in \mathbb{C}_+$ then $\exists k_\lambda \in K_\theta \; : < f, k_\lambda >= f(\lambda), \;\; \forall f \in K_\theta$:

$$k_\lambda(z) = \frac{1}{2\pi i} \frac{1 - \bar{\theta}(\lambda)\theta(z)}{\bar{\lambda} - z}.$$

Question:

For what Λ will $\{k_{\lambda_n}\}_{\lambda_n \in \Lambda}$ be complete in K_θ?

Extensions and Applications of BM Theory: Other completeness problems

Special functions

Consider the Schrödinger equation:

$$-u'' + qu = \lambda u, \quad \lambda \in \mathbb{C}$$

on (a, b), $q \in L^1_{loc}$, $a \neq -\infty$, q is summable near a. Fix a self-adjoint boundary condition at b. For each $\lambda \in C$ choose a solution u_λ. For $\Lambda = \{\lambda_n\} \in \mathbb{C}$ consider $U_\Lambda = \{u_{\lambda_n}\}$.

For different choices of q, U_Λ can be Bessel functions, Airy functions, etc.

Question

When is U_Λ complete in $L^2(a, b)$?

Toeplitz kernels

Let $\phi \in L^\infty(\mathbb{R})$. The Toeplitz operator T_ϕ with symbol ϕ is defined as

$$T_\phi : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+), \; T_\phi f = P_+ \phi f,$$

where P_+ is the orthogonal projection from $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$.

Completeness problem in K_θ in Toeplitz form:

$$\exists f \in K_\theta, \; f \perp \{k_{\lambda_n}\}_{\lambda_n \in \Lambda} \;\Leftrightarrow\; \exists f \in K_\theta, \; f = 0 \text{ on } \Lambda \; (f = B_\Lambda g) \;\Leftrightarrow$$

$$\exists g \in \ker T_{\bar\theta B_\Lambda} \;\; (\text{recall } \bar\theta f \in \bar H^2, \;\Leftrightarrow\; f \in K_\theta).$$

$$\{k_{\lambda_n}\}_{\lambda_n \in \Lambda} \text{ complete in } K_\theta \;\Leftrightarrow\; \ker T_{\bar\theta B_\Lambda} = \{0\}.$$

If $\Lambda \subset \mathbb{R}$, B_Λ is chosen so that $B_\Lambda = 1$ on Λ.

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If $\Lambda \subset \mathbb{R}$, B_Λ is chosen so that $B_\Lambda = 1$ on Λ.

Similarly, if $\{u_\lambda\}$ is a system of special functions $\{u_\lambda\}_{\lambda\in\Lambda} \subset L^2(a,b)$ corresponding to a Schrödinger equation with Weyl-Titchmarsh function θ, completeness $\Leftrightarrow$ injectivity of a Toeplitz operator. All in all, we have:

Lemma (N. Makarov, A. P.)

$\{k_\lambda\}$ is complete in K_θ iff $\{u_\lambda\}$ is complete in $L^2(a,b)$ iff $\ker T_{\bar\theta B_\Lambda} = \{0\}$.

BM problem is a problem of completeness of $\{k_{\lambda_n}\}_{\lambda_n\in\Lambda}$ in K_θ with $\theta = S = e^{iz}$.

BM Theorem in Toeplitz form: $\ker T_{\bar S^a B_\Lambda} \neq \{0\}$ if $D^*(\Lambda) < a$ and $= \{0\}$ if $< a$.

Question

How to determine if $\ker T_{\bar\theta B_\Lambda} = \{0\}$ for other θ?

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BM Theory for Toeplitz Kernels

Question

How to determine if $\ker T_Q = \{0\}$ for $Q = e^{i\gamma}$?

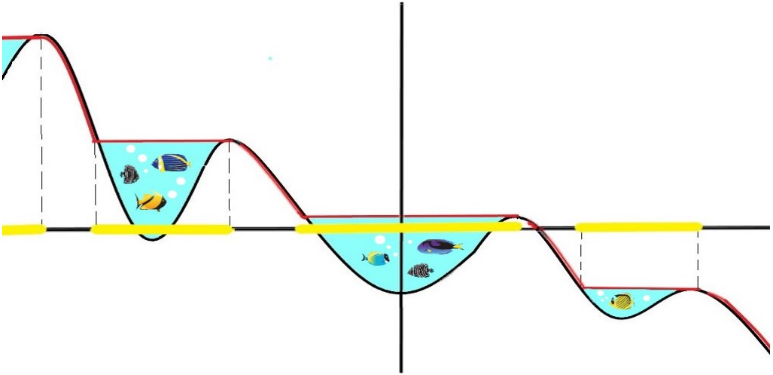
Let γ be a continuous function $\mathbb{R} \to \mathbb{R}$ such that $\gamma(\mp\infty) = \pm\infty$, i.e.,

$$\lim_{x \to -\infty} \gamma(x) = +\infty, \qquad \lim_{x \to +\infty} \gamma(x) = -\infty.$$

The family $BM(\gamma)$ is defined as the collection of the components of the open set

$$\left\{ x : \ \gamma(x) \neq \max_{[x,+\infty)} \gamma \right\}.$$

γ, γ^* and $BM(\gamma)$



For $\kappa \geq 0$, γ is (κ)-*almost decreasing* if
$\gamma(\mp\infty) = \pm\infty$ and $\sum_{I_n \in BM(\gamma)} \frac{|I_n|^2}{(\mathrm{dist}(0,I_n)+1)^{2-\kappa}} < \infty$.

BM Theory for Toeplitz Kernels

Given two smooth unimodular functions $U = e^{i\sigma}$ and $V = e^{i\gamma}$, we can consider the family of functions

$$\gamma_a = \arg(\bar{U}^a V) = \gamma - a\sigma, \qquad (a \in \mathbb{R}).$$

Define the transition parameter $c = c(U, V; \kappa)$ as

$$\inf\{a : \gamma_a \text{ is } (\kappa)\text{--almost descreasing}\}.$$

Theorem (N. Makarov, A.P.)

Let J be a meromorphic inner function, and suppose that a unimodular function U satisfies $(\arg\ U)'(x) \asymp |x|^\kappa$ as $x \to \infty$.
Let $c = c(U, J; \kappa)$. Then

$$\ker T_{\bar{U}^a J} = \{0\} \ \ \textit{for} \ \ a < c, \ \ker T_{\bar{U}^a J} \neq \{0\} \ \ \textit{for} \ \ a > c.$$

BM Theory for Toeplitz Kernels

Applications:

- $\kappa = 0 \quad \Leftrightarrow$ BM theorem $\leftrightarrow$ regular Schrödinger
- $\kappa = 1/2 \quad \leftrightarrow$ Airy case
- $\kappa = 1 \quad \leftrightarrow$ Harmonic oscillator
- etc.

Toeplitz form of BM and type problems:

BM problem: $\ker T_{\bar{S}^a\theta}$.

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Example: Airy Functions

Let $Ai(x)$ be the Airy function, a solution of $y'' = xy$ on $\mathbb{R}$ that is L^2 near $+\infty$. Fact: $Ai(z)$ is entire and

$$Ai(-\lambda) \asymp \lambda^{-1/4} \cos\left(\frac{2}{3}\lambda^{3/2} + c\right), \quad \lambda \to +\infty.$$

Problem

Given $\Lambda = \{\lambda_n\} \subset \mathbb{R}$, determine if $A_\Lambda = \{Ai(t - \lambda_n)\}$ is complete in $L^2(\mathbb{R}_+)$.

Solution

$Ai(t - \lambda) = u_\lambda(t)$ where u_λ is an L^2-solution of $-u'' + tu = \lambda u$ on $\mathbb{R}_+$. Up to a finite-dimensional gap, A_Λ is complete in $L^2(\mathbb{R}_+)$ iff $\ker T_{\bar{M}J_\Lambda} = \{0\}$, where $M = \exp(i\frac{\mathrm{sign} x+1}{3}x^{3/2})$ and J_Λ is any inner function satisfying $\{J_\Lambda = 1\} = \Lambda$.

The Type Problem

Let μ be a finite positive measure on the real line. For $a > 0$ denote by $\mathcal{E}_a$ the family of exponential functions

$$\mathcal{E}_a = \{e^{ist} |\ s \in [0, a]\}.$$

The exponential type of μ:

$$\mathcal{T}_\mu = \inf\{a > 0 |\ \mathcal{E}_a \text{ is complete in } L^2(\mu)\}$$

if the set of such a is non-empty and infinity otherwise.

Problem

Find $\mathcal{T}_\mu$ in terms of μ.

The L^p version, $\mathcal{T}^p_\mu$ for $1 < p \leq \infty$, can be defined similarly. The definition of type can be easily extended from finite to locally finite measures.

The Type Problem: History

This question first appears in the work of Wiener, Kolmogorov and Krein in the context of stationary Gaussian processes in 1930-40's. If μ is a spectral measure of a stationary Gaussian process, the property that $\mathcal{E}_a$ is complete in $L^2(\mu)$ is equivalent to the property that the process at any time can be predicted from the data for the time period from 0 to a.

The type problem can also be restated in terms of the Bernstein weighted approximation, see for instance Koosis' book. Connections with spectral theory of second order differential operators were studied by Gelfand, Levitan and Krein.

Known results

Krein (1945) proved that if $d\mu = w(x)dx$ and $\log w(x)/(1+x^2)$ is summable then $\mathcal{T}_\mu^p = \infty$ for all p. A partial inverse, proved by Levinson and McKean (1964), holds for even monotone w.

For discrete measures, in the case $\operatorname{supp}\mu = \mathbb{Z}$, a deep result by Koosis shows an analogue of Krein's result:

$$\text{if} \quad \mu = \sum w(n)\delta_n \quad \text{where} \quad \sum \frac{\log w(n)}{1+n^2} > -\infty,$$

then

$$\mathcal{T}_\mu^p = 2\pi$$

for all p. (How much can one perturb $\mathbb{Z}$ in this result?)

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Type Problem in Bernstein's form

A function $W : \mathbb{R} \to [1, +\infty]$ is called a weight if it is lower semicontinuous and tends to $+\infty$ at $\pm\infty$. Bernstein's space C_W is defined as a space of all continuous functions f on R such that $f = o(W)$ near $\pm\infty$ with the norm

$$||f||_W = ||f/W||_\infty.$$

Type of W:

$$\mathcal{T}_W = \inf\{a \,|\mathcal{E}_a \text{ is complete in } C_W\},$$

or ∞ if the set is empty.

Short partitions

Let

$$\ldots < a_{-2} < a_{-1} < a_0 = 0 < a_1 < a_2 < \ldots$$

be a two-sided sequence of real points. We say that the intervals $I_n = (a_n, a_{n+1}]$ form a short partition of $\mathbb{R}$ if $|I_n| \to \infty$ as $|n| \to \infty$ and the sequence $\{I_n\}$ is short, i.e., $\sum \frac{|I_n|^2}{1+\mathrm{dist}^2(I_n,0)} < \infty$.

D-uniform sequences

Let $\Lambda = \{\lambda_n\}$ be a sequence of distinct real points. We say that Λ is D-uniform if there exists a short partition I_n such that

$$\Delta_n = D|I_n| + o(|I_n|) \quad \text{as} \quad n \to \pm\infty \quad \textit{(density condition)}$$

and

$$\sum_n \frac{\Delta_n^2 \log|I_n| - L_n}{1 + \operatorname{dist}^2(0, I_n)} < \infty \quad \textit{(energy condition)}$$

where

$$\Delta_n = \#(\Lambda \cap I_n) \quad \text{and} \quad L_n = L(\Lambda \cap I_n) = \sum_{\lambda_k, \lambda_l \in I_n,\ \lambda_k \neq \lambda_l} \log|\lambda_k - \lambda_l|.$$

Type Theorem

In Bernstein's form:

Theorem

$$\mathcal{T}_W = 2\pi \sup \left\{ D \;\middle|\; \exists D - uniform\ \Lambda\ such\ that\ \sum \frac{\log W(\lambda_n)}{1+n^2} < \infty \right\}$$

In L^p form:

Corollary

Let μ be a finite positive measure on the line. Let $1 < p \leq \infty$ and $a > 0$ be constants. Then $\mathcal{T}^p_\mu \geq 2\pi a$ if and only if for any weight $W \in L^1(\mu)$ and any $0 < D < a$ there exists a D-uniform sequence $\Lambda = \{\lambda_n\} \subset \operatorname{supp} \mu$ such that

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Type Theorem

A shorter L^p form for finite types:

For a discrete sequence $\Lambda = \{\lambda_n\}$ denote by $\Lambda^* = \{\lambda_n{}^*\}$ the sequence of disjont intervals

$$\lambda_n{}^* = (\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n), \quad \varepsilon_n = \frac{1}{3}\mathrm{dist}(\lambda_n,\ \Lambda \setminus \{\lambda_n\}).$$

Theorem

Suppose that $\mathcal{T}_\mu < \infty$. Then

$$\mathcal{T}_\mu = 2\pi \max\left\{ D \,\middle|\, \exists\ D\text{-}uniform\ \Lambda = \{\lambda_n\}\ such\ that\ \sum \frac{\log \mu(\lambda_n{}^*)}{1+n^2} > -\infty \right\}$$

if the set of such d is non-empty and $\mathcal{T}_\mu = 0$ otherwise.

Examples of type: type alternative for Frostman measures

Recall that a positive measure μ on $\mathbb{R}$ is a Frostman measure if there exist positive constants α and C such that

$$\mu((x-\epsilon, x+\epsilon)) < C\epsilon^{\alpha} \tag{1}$$

for all $\epsilon > 0, x \in \mathbb{R}$.

Theorem

If μ is a Frostman measure then $\mathcal{T}_{\mu}$ equals either 0 or ∞.

In particular, all absolutely continuous measures $d\mu(x) = w(x)dx$ with $w \in L^p(\mathbb{R})$, $p > 1$ have types 0 or ∞

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Corollaries: discrete measures

Extension of Koosis' result:

Corollary

Let $B = \{b_n\} \subset \mathbb{R}$ be a discrete sequence and let μ be a finite positive measure supported on B,

$$\mu = \sum w(n)\delta_{b_n}.$$

Then the type of μ is equal to the supremum of all numbers D, such that B contaings a D-uniform subsequence $\{b_{n_k}\}$ satisfying

$$\sum_k \frac{\log w(n_k)}{1+k^2} > -\infty.$$

Corollaries: discrete measures

Extension of Koosis' result:

Interior BM density: $D_*(\Lambda) = \inf\{d \mid \exists \text{ long } \{I_n\} \text{ such that } \#(\Lambda \cap I_n) \leqslant d|I_n|\}, \ \forall n\}.$

Separated sequences: $\inf_{k \neq n} |\lambda_k - \lambda_n| > 0.$

Corollary

Let B be a separated sequence and let $w(n)$ be a bounded positive function of n satisfying

$$\sum w(n) < \infty \quad \textit{and} \quad \sum \frac{\log w(n)}{1 + n^2} > -\infty.$$

Then the measure

$$\mu = \sum w(n) \delta_{b_n}$$

satisfies

$$\mathcal{T}_\mu = 2\pi D_*(B).$$