Several 20+ year old problems about Banach spaces and operators on them

or

Some of what I have been doing this millenium*

Bill Johnson

* With a LOT of help!

Rio de Janeiro, August, 2018

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[J.J. Schäffer, 1976]: Is $D(X) = \infty$ for all infinite dimensional X?

The late V. Gurarii re-popularized this problem in the late 1990s and pointed out that $D(X) = \infty$ for many classes of spaces. Indeed, if dim $X = \infty$ and $E \subset X$ with dim $E < \infty$, then $\exists X_0$ s.t. $X \sim E \oplus_2 X_0$. Thus if $D(X)$ is finite, then X there is a constant C s.t.every finite dimensional space is C-isomorphic to a C-complemented subspace of X. This implies that X cannot have non trivial type or cotype or . . . . In particular, X cannot be any of the classical spaces or be super-reflexive.

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Call a Banach space X *K-elastic* provided every isomorph of X K-embeds into X. Call X *elastic* if X is K-elastic for some $K < \infty$.

Theorem.

If X is a separable Banach space so that for some K, every isomorph of X is K-elastic, then X is finite dimensional.

This implies that $D(X) = \infty$ if $\dim X = \infty$ and is separable. $C[0,1]$ is 1-elastic because every separable Banach space is isometrically isomorphic to a subspace of $C[0,1]$. We suspected that all separable elastic spaces contain isomorphs of $C[0,1]$ and remarked that our proof of the theorem could be streamlined a lot if this is true. We could not prove this but were able to use Bourgain's ℓ_∞ index theory to prove that a separable elastic space contains a subspace that is isomorphic to c_0 and used that in the proof of the theorem.

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Conjecture. (JO)

If X is an elastic infinite dimensional separable Banach space, then $C[0,1]$ is isomorphic to a subspace of X.

My third PhD student, D. Alspach, and B. Sari created a new index and used it to verify our conjecture [AS, JFA 2016]. Their proof is rather complicated, but even more recently [K. Beanland, R. Causey, Houston J.M. 2018] simplified the proof somewhat by using more descriptive set theory. It looks likely that the Alspach-Sari index will be used more down the road.

Schäffer's problem is open for non separable spaces. Under GCH there are 1-elastic universal spaces of every density character, but tools used in the separable setting are not available when the spaces are non separable. [G. Godefroy, 2010] proved that under Martin's Maximum Axiom Schäffer's problem has an affirmative answer for subspaces of ℓ_∞.

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An element T of a Banach algebra $\mathcal{A}$ is a commutator if $T = AB - BA$ for some A, B in $\mathcal{A}$.

If X is finite dimensional, the commutators in $\mathcal{L}(X)$ are the matrices with trace zero.

In a general unital Banach algegra $\mathcal{A}$, the only known obstruction to being a commutator is that I is not a commutator [Wintner, 1947], [Wielandt, 1949], which yields that $\lambda I + T$ is not a commutator if $\lambda \neq 0$ and T is in a (proper) ideal. So in $\mathcal{L}(X)$, $\lambda I + T$ is not a commutator if $\lambda \neq 0$ and T is a compact operator.

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Wielandt's proof that I is not a commutator:

If $I = AB - BA$ then by induction

$$\forall n \quad A^n B - BA^n = nA^{n-1}.$$

So A cannot be nilpotent and

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The same is true in $\mathcal{L}(\ell_p)$, $1 < p < \infty$ [C. Apostol, 1972] and in $\mathcal{L}(c_0)$ [Apostol, 1973].

Apostol gave some information about commutators in $\mathcal{L}(\ell_1)$ and $\mathcal{L}(\ell_\infty)$, while [Schneeberger, 1971] proved that compact operators in $\mathcal{L}(L_p)$, $1 < p < \infty$, are commutators, but no other characterizations of commutators in $\mathcal{L}(X)$ were given in the last millennium.

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In $\mathcal{L}(\ell_\infty)$, the unique maximal ideal is the set of *weakly* compact operators. [Dosev, J, 2009] proved that in $\mathcal{L}(\ell_\infty)$, all non commutators are of the form $\lambda I + T$ with $\lambda \neq 0$ and T weakly compact.

Assuming that $\mathcal{L}(X)$ has a unique maximal ideal, say $\mathcal{M}$, classifying the non commutators in $\mathcal{L}(X)$ as being of the form $\lambda I + T$ with $\lambda \neq 0$ and T in $\mathcal{M}$ involves two steps:

Step 1. Every operator $T \in \mathcal{M}$ is a commutator.

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The methods for proving **Step 1** in all cases where the complete classification of the commutators on the space X is known are based on the fact that if $T \in \mathcal{M}$ then for every subspace $Y \subset X$, $Y \simeq X$ and every $\varepsilon > 0$ there exists a complemented subspace $Y_1 \subseteq Y$, $Y_1 \simeq X$ such that $\|T_{|Y_1}\| < \varepsilon$. That is easy when $X = \ell_p$, $1 \le p < \infty$ or $X = c_0$ (where $\mathcal{M}$ is the set of compact operators), but harder on other spaces where it is known, such as ℓ_∞ or L_p, $1 \le p < \infty$. Even the "easy" cases rely on the fact that the space X admits a Pełczyński decomposition, which means that X is isomorphic (= linearly homeomorphic) to some "infinite direct sum" of itself: $X \simeq (\sum X)_p$, $1 \le p \le \infty$ or $p = 0$. (The left and right shifts on $(\sum X)_p$ help a lot!) In fact, if X is *any* space that admits a Pełczyński decomposition, then every compact operator on X is a commutator.

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Step 2. If $T \in \mathcal{L}(X)$ is not of the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$, then T is a commutator.

Assume that X admits a Pełczyński decomposition: $X \simeq (\sum X)_p$, $1 \leq p \leq \infty$ or $p = 0$ and $\mathcal{L}(X)$ has a unique maximal ideal $\mathcal{M}$.

Doing **Step 2** involves proving structural results on the Banach space X in order to apply the following

Theorem

[DJ, 2009] Let X be a Banach space that admits a Pełczyński decomposition. Let $T \in \mathcal{L}(X)$ be such that there exists a complemented subspace $Y \subset X$ such that $Y \simeq X$, $T_{|Y}$ is an isomorphism, $Y + T(Y)$ is complemented in X and $d(Y, T(Y)) > 0$. Then T is a commutator.

For L_p, $1 \leq p < \infty$, this was done in [DJS, 2010]. Here $\mathcal{M}$ is the set of operators T such that for any subspace Y of X with $Y \simeq X$, $T_{|Y}$ is not an isomorphism.

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Wild Conjecture. In $\mathcal{L}(X)$, all non commutators are of the form $\lambda I + T$ with $\lambda \neq 0$ and T in a proper ideal.

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Moreover, the following problem is open for *every* infinite dimensional space.

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Ideals in $L(L_p)$ [J, G. Pisier, G. Schechtman, 2018]

After C^* algebras, probably the most natural non commutative Banach algebras are the spaces of bounded linear operators on such classical Banach spaces as $L_p := L_p(0, 1)$.

In order to study any Banach algebra one must understand something about the closed ideals in the algebra. For, ℓ_p, $1 \leq p < \infty$, the only non trivial closed ideal is the ideal of compact operators. $L(L_p)$, $1 \leq p \neq 2 < \infty$, is much more complicated.

An ideal $\mathcal{I}$ is small if $\mathcal{I}$ is contained in the strictly singular operators. (An operator T is strictly singular if T is not an isomorphism when restricted to any infinite dimensional subspace.) Call $\mathcal{I}$ large if it is not small.

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Constructing ideals

Small ideal: Contained in the strictly singular operators.
Large ideal: Not small.

The closed ideal $\mathcal{I}_Y$ generated by a projection from X onto an infinite dimensional subspace Y is large if it is properly contained in $L(X)$. If Y is isomorphic to $Y \oplus Y$, $\mathcal{I}_Y$ is the closure of the operators on X that factor through Y and $\mathcal{I}_Y$ is proper if X is not isomorphic to a complemented subspace of Y. [Schechtman, 1975] proved that $L(L_p)$, $1 < p \neq 2 < \infty$, has at least $\aleph_0$ ideals by constructing $\aleph_0$ isomorphically different complemented subspaces of L_p. [Bourgain, Rosenthal, Schechtman, 1981] improved this to $\aleph_1$ by constructing $\aleph_1$ isomorphically different complemented subspaces of L_p.

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Building on other work this millenium, [Schlumprecht, Zsák, 2018] proved that $L(L_p)$, $1 < p \neq 2 < \infty$, has infinitely many–even a continuum–of closed small ideals, thereby solving in the process a problem in Pietsch's 1978 book "Operator Ideals". It remains open whether $L(L_p)$, $1 < p \neq 2 < \infty$, has more than a continuum of closed ideals.

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J, B. Maurey, and G. Schechtman, **BLMS** (2009)

[JMS, 2009] gives an affirmative answer to the question [Heinrich-Mankiewicz, 1982] whether the $\mathcal{L}_1$ spaces (direct limits of spaces that are uniformly isomorphic to finite dimensional L_1 spaces) are preserved under uniform equivalence (i.e., f, f^{-1} both uniformly continuous).

At the heart of the question is a recurring problem: Suppose a linear mapping $T : X \to Y$ admits a Lipschitz factorization through a Banach space Z; i.e., we have Lipschitz $F_1 : X \to Z$ and $F_2 : Z \to Y$ and $F_2 \circ F_1 = T$. What extra guarantees that T admits a linear factorization through Z?

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The main result in [JMS, 2009] is

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Let X be a finite dimensional normed space, Y a Banach space with the Radon Nikodym Property and $T : X \to Y$ a linear operator. Let Z be a separable Banach space and assume there are Lipschitz maps $F_1 : X \to Z$ and $F_2 : Z \to Y$ with $F_2 \circ F_1 = T$. Then for every $\lambda > 1$ there are linear maps $T_1 : X \to L_\infty(Z)$ and $T_2 : L_1(Z) \to Y$ with $T_2 \circ i_{\infty,1} \circ T_1 = T$ and $\|T_1\| \cdot \|T_2\| \leq \lambda \mathrm{Lip}(F_1)\mathrm{Lip}(F_2)$.

If Z is $\mathcal{L}_1$ then so is $L_1(Z)$ and hence T linearly factors through a $\mathcal{L}_1$ space. This and fairly standard tools in non linear geometric functional analysis give an affirmative answer to the problem from [HM, 1982].
The proof of the Theorem is based on a simple local-global linearization idea. For the application only the case where Y is finite dimensional is needed.

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J, Maurey, Schechtman, **JAMS 20** (2007)

The first weakly null normalized sequences (WNNS) with no unconditional subsequence were constructed in [Maurey, Rosenthal, 1977]. Their technique was incorporated into [Gowers, Maurey, 1992], but the examples in [MR, 1977] are still interesting because the ambient spaces were $C(K)$ with K countable, which are hereditarily c_0. Every subsequence of the WNNS they constructed reproduces the (conditional) summing basis on blocks.

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J and B. Zheng, A characterization of subspaces and quotients of reflexive Banach spaces with unconditional basis; I & II, **Duke M.J. 141** (2008), **Israel J.M. 185** (2011)

Problem from the 1970s: Give an intrinsic characterization of Banach spaces that embed into a space that has an unconditional basis.

Every space with an unconditional expansion of the identity (in particular, every space with an unconditional finite dimensional decomposition) embeds into a space with unconditional basis [Pełczyński, Wojtaszczyk, 1971], [Lindenstrauss, Tzafriri, 1977].

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The only apparent useful invariant is that in a subspace of a space with unconditional basis, every WNNS has an unconditional basic sequence. A quotient of a space with shrinking unconditional basis has this property [J, 1977], [Odell, 1986].

Also, a reflexive quotient X of a space with shrinking unconditional basis embeds into a space with unconditional basis as long as X has the approximation property [M. Feder, 1980].

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So there were two problems

1. Give an intrinsic characterization of Banach spaces that embed into a space that has an unconditional basis.

2. Does every quotient of a space with shrinking unconditional basis embed into a space with unconditional basis?

Much research centered around reflexive spaces. Every reflexive subspace of a space with unconditional basis embeds into a reflexive space with unconditional basis [Davis, Figiel J, Pełczyński, 1974], [Figiel, J, Tzafriri, 1975].

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In [JZ, 2008] both problems are given affirmative answers for reflexive spaces. [JZ, 2011] gives an affirmative answer to (2) in general and to (1) for spaces that have a separable dual.

The answers for reflexive spaces follow from the following theorem:

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Let X be a separable reflexive Banach space. Then the following are equivalent.

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The answers for reflexive spaces follow from the following theorem:

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Let X be a separable reflexive Banach space. Then the following are equivalent.

(a) X has the UTP.

(b) X is isomorphic to a subspace of a Banach space with an unconditional basis.

(c) X is isomorphic to a subspace of a reflexive space with an unconditional basis.

(d) X is isomorphic to a quotient of a Banach space with a shrinking unconditional basis.

(e) X is isomorphic to a quotient of a reflexive space with an unconditional basis.

(f) X is isomorphic to a subspace of a quotient of a reflexive space with an unconditional basis.

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(i) X is isomorphic to a quotient of a reflexive subspace of a Banach space with a shrinking unconditional basis.

(i) X^* has the UTP.

Definition.

[Odell-Schlumprecht] A branch of a tree is a maximal linearly ordered subset of the tree under the tree order. X has the C-unconditional tree property (C-UTP) if every normalized weakly null infinitely branching tree in X has a C-unconditional branch. X has the UTP if X has the C-UTP for some $C > 0$.

The UTP is a strengthening of the property “every WNNS has an unconditional subsequence". The weaker property for a reflexive space does NOT imply embeddability into a space with unconditional basis [JZ, 2008].

The proof of the theorem uses some new tricks, blocking methods developed in the 1970s, and the analysis in [Odell-Schlumprecht, 2002, 2006] relating tree properties to embeddability into spaces that have a finite dimensional decomposition with the corresponding skipped blocking property.

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For Banach spaces with a separable dual, there is a similar theorem [JZ, 2011], but the characterization involves the weak* UTP. A Banach space X is said to have the weak* UTP provided every normalized weak* null tree in X^* has a branch that is an unconditional basic sequence. The main new technical feature in [JZ, 2011] is that blocking and "killing the overlap" techniques originally developed for finite dimensional decompositions are adapted to work for blockings of shrinking M-bases (that is, biorthogonal sequences $\{x_n, x_n^*\}$ with span x_n dense in X and span x_n^* dense in X^*). Shrinking M-bases are known to exist in every Banach space that has a separable dual. These technical advances provide some simplifications of the argument in the reflexive case presented in [JZ, 2009] and likely will be used in the future to study the structure of Banach spaces which do not have a basis or a finite dimensional decomposition.

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J, A. Nasseri, Schechtman, T. Tkocz, **Canad. Math. Bull. 58** (2015)

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Tauberian operators

Nasseri's problem is related to Tauberian operators on $L_1 := L_1(0,1)$. An operator $T : X \to Y$ is called *Tauberian* if $T^{**-1}(Y) = X$ [Kalton, Wilansky, 1976]

The book [M. Gonzáles and A. Martínez-Abejón, 2010] on Tauberian operators contains:

Theorem

Let $T : L_1(0,1) \to Y$. The following are equivalent.

0. *T is Tauberian.*
1. *For all normalized disjoint sequences $\{x_i\}$, $\liminf_{i\to\infty} \|Tx_i\| > 0$.*
2. *If $\{x_i\}$ is equivalent to the unit vector basis of ℓ_1 then there is an N such that $T_{|[x_i]_{i=N}^{\infty}}$ is an isomorphism.*
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$T : X \to Y$ is Tauberian: $T^{**-1}(Y) = X$.

What is the connection between Tauberian operators on L_1 and dense range, non surjective operators on ℓ_∞?

If T is 1-1 Tauberian, T^{**} is 1-1.

Thus, if T is a Tauberian operator on L_1 that is 1-1 but does not have closed range, then T^* is a dense range operator on L_∞ that is not surjective.

L_∞ is isomorphic to ℓ_∞, so having a 1-1 Tauberian, non closed range operator on L_1 gives a positive answer to Nasseri's question.

In fact, whether there is such an operator on L_1 is *a priori* equivalent to Nasseri's question.

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One of the main open problems from [Weis, Wolff, 1984] mentioned in [GM, 2010] is:

Is there a Tauberian operator T on L_1 whose kernel is infinite dimensional?

If T satisfies this condition, then you can play around and get a perturbation S of T that is Tauberian, 1-1, and has dense, non closed range (so is not surjective). Taking the adjoint of S and replacing L_∞ by its isomorph ℓ_∞, you would have a 1-1, dense range, non surjective operator on ℓ_∞.

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Bottom line: The question whether there is a dense range non surjective operator on the non separable space ℓ_∞ is really a question about the existence of a Tauberian operator with infinite dimensional kernel on the separable space L_1 .

Theorem: [G, M-A, 2010] Let $T : L_1(0,1) \to Y$. TFAE

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T satisfying condition (3) and having an infinite dimensional kernel has a known finite dimensional analogue:

CS Theorem [Berinde,Gilbert,Indyk,Karloff,Strauss, 2008]: For each n sufficiently large putting $m = [3n/4]$, there is an operator $T : \ell_1^n \to \ell_1^m$ such that

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J, A. Szankowski, Complementably universal Banach spaces, II. **JFA 257** (2009)

Given a family $\mathcal{F}$ of operators between Banach spaces, it is natural to try to find a single (usually separable) Banach space Z s.t. all the operators in $\mathcal{F}$ factor through Z. If $\mathcal{F}$ is the collection of all operators between separable Banach spaces that have the bounded approximation property, there are such separable Z [Pełczyński, 1969], [M. Kadec, 1971], [Pełczyński, 1971]. These, and even some reflexive spaces [J, 1971], have the property that every operator that is uniformly approximable by finite rank operators factor through Z. There is not a separable space s.t. every operator between separable spaces factors through it [J, Szankowski, 1976], but this paper left open the possibility that there is a separable space s.t. every compact operator factors through it. It turns out that no such space exists [JS, 2009].

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J, Szankowski, Some more Banach spaces that have the hereditary approximation property, **Annals Math. 176** (2012)

A Banach space has the hereditary approximation property (HAP) provided every subspace has the approximation property. There are non Hilbertian spaces that are HAPpy [J, 1980], [Pisier, 1988]. All of these examples are asymptotically Hilbertian; i.e., for some K and every n, there is a finite codimensional subspace all of whose n-dimensional subspaces are K-isomorphic to ℓ_2^n. An asymptotically Hilbertian space must be superreflexive and cannot have a symmetric basis unless it is isomorphic to a Hilbert space. This led to two problems [J, 1980]:

1. Can a non reflexive space have the HAP?
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The HAP is very difficult to work with. It does not have good permanence properties–there are spaces X and Y that have the HAP s.t. $X \oplus Y$ fails the HAP [Casazza, Garcia, J, 2001]. The main result of [J, Szankowski2, 2012] gives an affirmative answer to problem 2 from [J, 1980]:

Theorem.

There is a function $f(n) \uparrow \infty$ s.t. if for infinitely many n we have $D_n(X) \le f(n)$, then X has the HAP.

Here $D_n(X) := \sup d(E, \ell_2^n)$, where the sup is over all n-dimensional subspaces of X.

You can build Banach spaces with a symmetric basis even Orlicz sequence spaces that are not isomorphic to a Hilbert space and yet $D_n(X)$ goes to infinity as slowly as is desired. Hence problem (2) has an affirmative answer.

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T. Figiel, J, Pełczyński, **Israel J. Math.** 183 (2011)

It was a privilege for Tadek Figiel and me to be co-authors on A. Pełczyński's last paper. The solution to a (not especially important) problem that had eluded Tadek and me [Figiel, J, 1973] just dropped out, so I have an excuse to include a discussion of part of [FJP, 2011] in this talk.

Joint BAP for a Banach space and a subspace

Let X be a Banach space, let $Y \subseteq X$ be a subspace, let $\lambda \geq 1$. The pair (X, Y) is said to have the λ-BAP if for each $\lambda' > \lambda$ and each subspace $F \subseteq X$ with $\dim F < \infty$, there is a finite rank operator $u : X \to X$ such that $||u|| < \lambda'$, $u(x) = x$ for $x \in F$ and $u(Y) \subseteq Y$.

If (X, Y) has the λ-BAP then X/Y has the λ-BAP. Thus by [Szankowski, 2009], for $1 \leq p < 2$ there are subspaces Y of ℓ_p that have the BAP and yet (ℓ_p, Y) fails the BAP.

It is open whether (X, Y) has the BAP if X, Y, and X/Y all have the BAP, but I don't believe it.

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It is open whether (X, Y) has the BAP if X, Y, and X/Y all have the BAP, but I don't believe it.

If Y is a finite dimensional subspace of X and X has the λ-BAP then also (X, Y) has the λ-BAP and hence also X/Y has the λ-BAP. That is, the λ-BAP passes to quotients by finite dimensional subspaces. By duality you get that if X^* the λ-BAP then every finite codimensional subspace of X has the λ-BAP. In particular, every finite codimensional subspace of an L_1 space has the 1-BAP. Easy as this is, I don't think that anyone previously had noticed this.

In fact,

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X^* has the λ-BAP iff (X, Y) has the λ-BAP for every finite codimensional subspace Y.

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The following proposition turned out to be useful.

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Let X be a Banach space and let $Y \subseteq X$ be a closed subspace such that $\dim X/Y = n < \infty$ and Y has the $\lambda-$BAP. Then the pair (X, Y) has the $3\lambda-$BAP.

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If X is a Banach space and Y has the $\lambda-$BAP for every finite codimensional subspace $Y \subseteq X$, then X^* has the $3\lambda-$BAP.

Consequently, in contradistinction to the case of commutative L_1 spaces, for every λ there are finite codimensional subspaces Y of the non commutative L_1 space S_1 of trace class operators on ℓ_2 that fail the λ-BAP because $L(\ell_2)$ fails the AP [Szankowski, 1981] and $L(\ell_2)$ is the dual to S_1.

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The main result [FJ, 1972] is that there is a subspace of c_0 that has the AP but fails the BAP. We could not prove the same result for ℓ_1.

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[FJP, 2011] There is a subspace Y of ℓ_1 that has the AP but fails the BAP.

Proof. Start with a subspace X of ℓ_1 that fails the approximation property [Szankowski, 1978]. From the existence of such a space it follows [J, 1972] that if we let Z be the ℓ_1–sum of a dense sequence (X_n) of finite dimensional subspaces of X, then Z^* fails the BAP and yet Z has the BAP. Then Y can be the ℓ_1– sum of a suitable sequence of finite codimensional subspaces of Z by the previous Corollary..

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Thank you for your attention