Bourgain–Delbaen $\mathscr{L}_\infty$-spaces, the Scalar-plus-Compact property and Invariant Subspaces

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International Congress of Mathematicians, Rio de Janiero, August 2018

Introduction

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The BD construction has also been useful in establishing new positive results (that is to say *"not just counterexamples"*), notably the embedding theorem of Freeman–Odell–Schlumprecht, and more recently in finding new proofs of some important theorems of Zippin. Again, time constraints will prevent us from looking at these developments in as much detail as we should wish.

Plan

The plan for the rest of the talk is as follows:

- $\mathscr{L}_\infty$-spaces and the general BD-construction
- Standard BD-spaces
- The FOS embedding theorem
- Tsirelson spaces and "few-operators" constructions
- The space $\mathfrak{X}_{\mathrm{K}}$ with very few operators
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More precisely, a Banach space X is a $\mathscr{L}_\infty$-space if there is a constant M such that, for every finite-dimensional subspace E of X, there is a finite-dimensional subspace F with $E \subseteq F \subseteq X$ that is M-isomorphic to a finite-dimensional ℓ_∞-space.

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So a separable space X is $\mathscr{L}_{\infty,M}$ iff there is an increasing sequence of finite-dimensional subspaces $E_1 \subseteq E_2 \subseteq \cdots$ such that $\bigcup_{n\in\mathbb{N}} E_n$ is dense in X and E_n is M-isomorphic to $\ell_\infty^{\dim E_n}$ for each n.

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We are going to investigate a particular way to construct $\mathscr{L}_\infty$-spaces, based on *successive linear extensions*

A little notation

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There is a natural duality between the spaces $\mathbb{R}^\Gamma$ and $\mathbb{R}^{(\Gamma)}$, given by $\langle f, x \rangle = \sum_{\gamma \in \Gamma} f(\gamma) x(\gamma)$, and generally we shall think of elements of $\mathbb{R}^\Gamma$ as *vectors*, while elements of $\mathbb{R}^{(\Gamma)}$ are seen as *functionals*.

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In particular, the function that takes the value 1 at γ, while vanishing elsewhere, will be denoted e_γ if we think of it as the usual unit vector, and e^*_γ if we think of it as the *evaluation functional* $x \mapsto x(\gamma)$.

Successive extension mappings (easy linear algebra)

Let $(\Gamma_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite sets with union $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$, and introduce the difference sets $\Delta_1 = \Gamma_1$, $\Delta_{n+1} = \Gamma_{n+1} \setminus \Gamma_n$. Suppose that, for each n and each $\gamma \in \Delta_{n+1}$, we specify some $c^*_\gamma \in \mathbb{R}^{\Gamma_n}$. For completeness, set $c^*_\gamma = 0$ when $\gamma \in \Gamma_1$.

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We can define, for each n, a *linear extension mapping* $i_{n+1,n} : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^{\Gamma_{n+1}}$, by

$$i_{n+1,n}u(\gamma) = \begin{cases} u(\gamma) & \text{if } \gamma \in \Gamma_n \\ \langle c^*_\gamma, u\rangle & \text{if } \gamma \in \Delta_{n+1}. \end{cases}$$

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By forming compositions $i_{m,m-1} \circ i_{m-1,m-2} \circ \cdots \circ i_{n+1,n}$ and letting m increase, we obtain, for each n, a linear extension mapping $i_n : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^{\Gamma}$. Writing r_n for the restriction mapping $\mathbb{R}^{\Gamma} \to \mathbb{R}^{\Gamma_n}$, we have the extension condition $r_n i_n r_n = r_n$ and the compatibility condition $i_m r_m i_n = i_n$ for $n < m$.

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The images $F_n = i_n[\mathbb{R}^{\Gamma_n}]$ form an increasing sequence of finite-dimensional subspaces of $\mathbb{R}^\Gamma$. For each n, $P_n = i_n r_n$ is a projection on $\mathbb{R}^\Gamma$ with image F_n.

Left- and right-triangular bases

Continuing from the last slide, we introduce vectors d_γ and functionals d_γ^* defined as follows

$$\begin{aligned} d_\gamma &= i_n e_\gamma && \text{when } \gamma \in \Delta_n \\ d_\gamma^* &= e_\gamma^* - c_\gamma^* \end{aligned}$$

Observation

- $(d_\gamma^*)_{\gamma\in\Gamma}$ is an algebraic basis of $\mathbb{R}^{(\Gamma)}$
- $(d_\gamma)_{\gamma\in\Gamma}$ is the unique system in $\mathbb{R}^\Gamma$ that is biorthogonal to $(d_\gamma^*)_{\gamma\in\Gamma}$, and is an algebraic basis of $F = \bigcup_n F_n$
- the projection $P_n = i_n r_n : \mathbb{R}^\Gamma \to F_n$ is given by $P_n x = \sum_{\gamma\in\Gamma_n} \langle d_\gamma^*, x\rangle d_\gamma$
- the dual projection P_n^* takes $\mathbb{R}^{(\Gamma)}$ onto $\mathbb{R}^{\Gamma_n}$ and is given by $P_n^* f^* = \sum_{\gamma\in\Gamma_n} \langle f^*, d_\gamma\rangle d_\gamma^*$

The general BD construction

Suppose that in the preceding discussion the extension operators i_n turn out to take values in $\ell_\infty(\Gamma)$ with a uniform bound $\|i_n\| \le M$ for the norm of i_n as an operator from $\ell_\infty(\Gamma_n)$ to $\ell_\infty(\Gamma)$.

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- For each n, F_n is M-isomorphic to $\ell_\infty(\Gamma_n)$, and hence the subspace X of $\ell_\infty(\Gamma)$, defined as $X = \overline{\bigcup_{n\in\mathbb{N}} F_n}$, is $\mathscr{L}_{\infty,M}$
- The functionals $d^*_\gamma = e^*_\gamma - c^*_\gamma$ form a Schauder basis of $\ell_1(\Gamma)$
- The biorthogonal vectors $d_\gamma \in \ell_\infty(\Gamma)$ form a Schauder basis of X
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What we have just described is the *General BD Construction* and the resulting space X is what we call a (general) *BD-space.* The set Γ, equipped with the compatible sequence of extension mappings (i_n), or equivalently with the family of functionals (c^*_γ), is what we call a BD-set.

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Although the use of extension operators might seem to limit the scope of constructions of this kind, Argyros, Gasparis and Motakis recently showed that every separable $\mathscr{L}_\infty$-space (and, in particular, every isomorphic predual of ℓ_1) is isomorphic to a BD-space.

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What is missing at this level of generality is a useful condition on the functionals c^*_γ that would guarantee uniform boundedness of the extensions i_n. It would also be nice to have a criterion for the canonically embedded isomorphic copy of $\ell_1(\Gamma)$ to make up the whole of X^*.

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We say that Γ is a *Standard BD-set* with *weight* β if, for all n and all $\gamma \in \Delta_{n+1}$, the functional c_γ^* has one of the forms

$$c_\gamma^* = \qquad \beta(I - P_s^*)b^*$$

with $0 \le s \le n$ and $b^* \in \operatorname{ball} \ell_1(\Gamma_n \setminus \Gamma_s)$, or

$$c_\gamma^* = e_\xi^* + \beta(I - P_s^*)b^*$$

with $1 \le s \le n$, $\xi \in \Gamma_s$ and $b^* \in \operatorname{ball} \ell_1(\Gamma_n \setminus \Gamma_s)$.

Boundedness for standard BD-spaces

A crucial but easy estimate essentially due to B and D

Let Γ be a standard BD set of weight β. Then the extension operators i_n are uniformly norm bounded by $M = (1 - 2\beta)^{-1}$ and $X(\Gamma)$ is a $\mathscr{L}_{\infty,M}$-space.

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We then call $X(\Gamma)$ a standard BD-space.

Standard terminology for standard BD-spaces

Recall that c_γ^* is to be either $\beta(I - P_s^*)b^*$ or $e_\xi^* + \beta(I - P_s^*)b^*$. We call ξ (when it exists) the *base* of γ, b^* the *top* and s the *cut*.

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So to define a standard BD-structure on a set Γ we need first to express it as an increasing union of finite subsets Γ_n and then to define functions

$$\text{base} : \Gamma \to \Gamma \cup \{\text{undefined}\}, \quad \text{top} : \Gamma \to \text{ball}\, \ell_1(\Gamma), \quad \text{cut} : \Gamma \to \mathbb{N}$$

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Recall that c^*_γ is to be either $\beta(I - P^*_s)b^*$ or $e^*_\xi + \beta(I - P^*_s)b^*$. We call ξ (when it exists) the *base* of γ, b^* the *top* and s the *cut*.

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Duality for standard BD-spaces

A standard BD-set Γ has a *tree-structure* in which the immediate successors of ξ are the elements γ with base $\gamma = \xi$. This is intimately related to duality issues.

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Theorem

Let Γ be a standard BD-set. Then the dual $X(\Gamma)^*$ is naturally identifiable with $\ell_1(\Gamma)$ if and only if there is no infinite sequence (ξ_n) in Γ with $\xi_n = \text{base}\, \xi_{n+1}$ for all n, that is to say, if and only if the tree structure on Γ is well-founded.

Duality for standard BD-spaces

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When Γ is well-founded, it has a countable *ordinal index* that becomes important when we look at the finer structure of $X(\Gamma)$.

Plan

- $\mathscr{L}_\infty$-spaces and the general BD-construction
- Standard BD-spaces
- The FOS embedding theorem
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- The space $\mathfrak{X}_K$ with very few operators
- The Invariant Subspace Property

The embedding theorem of Freeman, Odell and Schlumprecht

It is an elementary fact that every separable Banach space is a quotient of ℓ_1. The surprising theorem of FOS is that when Y^* is a separable *dual space* then the quotient operator can itself be taken to be the dual of an embedding $Y \to X$ where X is an isomorphic predual of ℓ_1.

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Theorem [FOS (2011)]

Let Y be a Banach space with separable dual Y^*. Then, for each $0 < \epsilon < \frac{1}{2}$, there exists a well-founded special BD-set Γ and a $(1+\epsilon)$-isomorphic embedding $Y \to X(\Gamma)$.

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One approach to proving this theorem involves a careful construction of a certain ϵ-net Γ in ball Y^*, which we then equip with a standard BD structure in such that the natural map $J : Y \to \ell_\infty(\Gamma); (Jy)(\gamma) = \gamma(y)$ takes values in $X(\Gamma)$.

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He defined a norm on $\mathbb{R}^{(\mathbb{N})}$ by

$$\|x\|_T = \sup_{f^* \in W} |\langle f^*, x\rangle|,$$

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where the *norming set* $W \subset \mathbb{R}(\mathbb{N})$ is taken to be the smallest convex set such that $\pm e_n^* \in W$ for all n, and

$$\tfrac{1}{2}(f_1^* + f_2^* + \cdots + f_n^*) \in W \quad \text{whenever} \quad \begin{array}{l} f_1^*, \ldots, f_n^* \in W, \\ \max \operatorname{supp} f_k^* < \min \operatorname{supp} f_{k+1}^* \quad (1 \le k < n) \\ \text{and} \quad n \le \min \operatorname{supp} f_1^*. \end{array}$$

The space T is the completion of $\mathbb{R}^{(\mathbb{N})}$ for this norm.

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Schlumprecht's arbitrarily distortable space S (1991) is a space of this type, and its discovery was a necessary step towards the celebrated counterexamples of Gowers (1993-4) and the Gowers–Maurey theory of hereditarily indecomposable (HI) Banach spaces.

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Typically, we take a sequence of weights β_i tending rapidly to 0, for instance $\beta_i = 2^{-2^i}$, and a rapidly increasing sequence of regular families $\mathscr{M}_i$. The resulting space is denoted $T[(\beta_i, \mathscr{M}_i)_{i \in \mathbb{N}}]$ which we abbreviate to T_{mixed}.

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This is not the case for mixed Tsirelson spaces, though typically there is a class of special sequences (called RIS) for which T_{mixed}-upper estimates hold. This class is rich in the sense that every infinite-dimensional subspace contains a RIS.

Few operators

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A typical construction of a space with few operators would start with the ingredients $(\beta_i, \mathscr{M}_i)_{i \in \mathbb{N}}$ of a mixed Tsirelson space, but add additional restrictions on the functionals f_k^* that are allowed in the creation of new elements

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Using a coding method that goes back to Maurey and Rosenthal (1977) this can be done in such a way that for every operator T on the resulting Banach space X there is a scalar λ such that $\|Tx_n - \lambda x_n\| \to 0$ for every RIS (x_n).

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Our quest for a space with very few operators is going to bring us back to the BD-construction, and the first step will be to connect standard BD-spaces to Tsirelson norms.

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The evaluation analysis

Let Γ be a well-founded standard BD set of weight β and let γ be an element of Γ. Recall that $c_\gamma^* = e_\xi^* + \beta(I - P_s^*)b^*$ if $\xi = \text{base}\,\gamma$ is defined.

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$$e^*_\gamma = \sum_{j=1}^{a} \left(\beta(I - P^*_{s_j})b^*_j + d^*_{\xi_j}\right)$$

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A basic inequality

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$$e_\gamma^* = \sum_{j=1}^{a} \left(\beta(I - P_{s_j}^*)b_j^* + d_{\xi_j}^*\right)$$

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I have cheated slightly: sometimes $\mathscr{M}$ needs to be *slightly* bigger than this.

BD-sets with mixed weights

The next step is to generalize our notion of standard BD-space to allow different elements to have different weights, drawn from a suitable sequence (β_i) but subject to the condition $\text{weight}\,\gamma = \text{weight}\,\text{base}\,\gamma$ when this makes sense.

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Crucial Lemma

Let Γ be a well-founded standard BD-space with mixed weights, let Z be any Banach space and let $S : X(\Gamma) \to Z$ be a bounded linear operator. If $\|Sx_n\| \to 0$ for every RIS then $\|Sx_n\| \to 0$ for every bounded, weakly null sequence, and hence S is compact.

Constructing $\mathfrak{X}_{\mathrm{K}}$

To construct $\mathfrak{X}_{\mathrm{K}}$, we fix in advance a suitable fast-decreasing sequence of weights β_i and a fast-increasing sequence of regular families $\mathscr{M}_i$. We then use recursion to build a standard BD-set Γ^{K} with mixed weights, making sure as we add elements of Δ_{n+1} at stage $n+1$ that the history of any element of weight β_i is in the family $\mathscr{M}_i$, but also using a coding to restrict the choice of top γ when γ has weight β_i with i odd.

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In a way that is not too different from earlier mixed-Tsirelson constructions, we show that the space $\mathfrak{X}_K = X(\Gamma^K)$ is HI and that for every operator T on $\mathfrak{X}_K$ there is a scalar λ such that $\|(T-\lambda I)x_n\| \to 0$ for every RIS.

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Theorem

The space $\mathfrak{X}_{\mathrm{K}}$ has the Scalar-plus-Compact Property.

Tarbard's spaces

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It seemed natural to ask whether an isomorphic predual of ℓ_1 with few operators automatically has very few.Tarbard answered this question in a striking way.

Theorem [Tarbard (2012)]

For each natural number d there is a well-founded BD-set Γ^d and a strictly singular operator S on $X(\Gamma^d)$ such that

- $S^{d+1} = 0$
- every bounded linear operator T on $X(\Gamma^d)$ is uniquely expressible as

$$T = \lambda_0 I + \sum_{r=1}^{d} \lambda_r S^r + K,$$

Embeddings into spaces with very few operators

We finish this part of the talk with two results that give partial answers to the question: “For which Banach spaces Y does there exist an isomorphic embedding of Y into a space with very few operators?”. It turns out that the answer is affirmative for some spaces that are in themselves not at all pathological.

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We note that this result applies in particular when Y is a separable Hilbert space. We conjecture that for spaces Y with separable dual, the only obstruction to embeddability in a space with very few operators is the existence of a subspace of Y isomorphic to c_0. While we have some improvements of the above theorem, we cannot prove this "best possible" result.

Spaces with very few operators and non-separable dual

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Theorem [S.A.–R.H.–Th. Raikoftsalis]

There is a standard BD-set Γ (not well-founded!) such that $X(\Gamma)$ contains ℓ_1 and has very few operators.

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The Invariant Subspace Problem

Definition

Let X be a Banach space and T a bounded linear operator on X. The operator T admits a non-trivial invariant subspace if there exists a closed subspace Y of X with $Y \neq \{0\}$, $Y \neq X$ such that $T[Y] \hookrightarrow Y$.

The Invariant Subspace Problem

Problem

Let X be an infinite dimensional separable Banach space and $T : X \to X$ a bounded linear operator. Does T admit a non trivial closed invariant subspace?

- The above problem remains open in the case of the separable Hilbert space. However there are many results concerning certain classes of operators and Banach spaces.

The Invariant Subspace Problem

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Let X be an infinite dimensional separable Banach space and $T : X \to X$ a bounded linear operator. Does T admit a non trivial closed invariant subspace?

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Positive results

- In 1954 M. Aronszajn and K. T. Smith proved that every compact operator defined on an infinite dimensional Banach space admits a non trivial closed invariant subspace.
- In 1973 V. I. Lomonosov generalized the above result, by showing that in a complex Banach space X, every bounded linear operator $T \neq \lambda I$, for $\lambda \in \mathbb{C}$, which commutes with a non zero compact operator, admits a non trivial closed hyperinvariant subspace.
- N. D. Hooker (1981) and G. Sirotkin (2005) presented a version of the above theorem for real Banach spaces. In particular, an operator T defined on a real infinite dimensional Banach space, which satisfies the assumptions of Lomonosov's Theorem, admits a non trivial closed hyperinvariant subspace if and only if $(\alpha I - T)^2 + \beta^2 I \neq 0$, for every $\alpha, \beta \in \mathbb{R}$, with $\beta \neq 0$.

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Negative results

- P. Enflo and C. J. Read, in papers published in 1987 and 1984 respectively, present the first examples of operators on non reflexive, non classical Banach spaces, not admitting a non trivial closed invariant subspace.
- Also, C. J. Read proved that any separable Banach space containing either c_0 or complemented ℓ_1, admits an operator without a non trivial closed invariant subspace.
- Since c_0 is separably injective, c_0 is a complemented subspace of X. The assumption that ℓ_1 is a complemented subspace is necessary by our result with Raikoftsalis, mentioned earlier.
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Banach spaces with the Invariant Subspace property

- A Banach space X is said to have the Invariant Subspace Property (ISP), if every bounded linear operator on X admits a non trivial closed invariant subspace. A Banach space satisfies the hereditary ISP if every closed subspace of it satisfies ISP.
- Any Banach space with the scalar-plus-compact property, satisfies the Invariant Subspace Property.
- The first example of a Banach space satisfying ISP is the $\mathcal{L}_\infty$ space in the joint work with R. Haydon, mentioned before. However, it is not known if the space satisfies the hereditary ISP.

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A reflexive space with ISP

- The scalar-plus-compact problem for reflexive spaces remains open. Moreover, there is no known example of a bounded linear operator acting on a reflexive Banach space with no non trivial closed invariant subspace.

- In the case of reflexive spaces every operator is weakly compact. We know from the classical Aronszajn - Smith theorem that compact operators admit non-trivial invariant subspaces. The question is if the weak compactness ensures the same property for weakly compact operators.

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A reflexive space with ISP

- Recently, in a joint work with Pavlos Motakis, we have constructed a reflexive Banach space with ISP. Furthermore, this is the first example of a Banach space, with hereditary ISP.

A reflexive space with ISP

- More precisely, a reflexive HI Banach space $\mathfrak{X}_{\text{ISP}}$ is constructed satisfying the following.

(i) For every Y closed subspace of $\mathfrak{X}_{\text{ISP}}$, every $T \in \mathcal{L}(Y)$ is of the form $T = \lambda I + S$, with S strictly singular.

(ii) For every Y and $S, T : Y \to Y$ strictly singular operators, the composition ST is a compact one.

(iii) For every infinite dimensional subspace Y of $\mathfrak{X}_{\text{ISP}}$, the space $S(Y)$ is non-separable.

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- Every Banach space satisfying (i) and (ii), also satisfies the hereditary ISP.
- Indeed, for every non zero strictly singular S, either $S^2 = 0$ and $\ker S$ is a non trivial closed hyperinvariant subspace for S, or S commutes with S^2, which is non zero compact. In the latter case Lomonosov-Sirotkin Theorem yields that S admits a non trivial closed hyperinvariant subspace.
- The Calkin Algebra $\mathcal{C}(\mathfrak{X}_{\mathrm{ISP}}) := \mathcal{L}(\mathfrak{X}_{\mathrm{ISP}})/\mathcal{K}(\mathfrak{X}_{\mathrm{ISP}})$ is non-separable and for $S, T \in S(\mathfrak{X}_{\mathrm{ISP}})$, $[T][S] = 0$. This yields that $\mathcal{C}(\mathfrak{X}_{\mathrm{ISP}})$ admits at least two non equivalent complete multiplicative norms (Skillicorn).

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Spreading Models

Let $\{x_k\}_k$ be a Schauder basic sequence in a Banach space.

(i) We say that $\{x_k\}_k$ generates c_0 as a spreading model, if there exists a constant $C > 0$, such that for any $n \leqslant k_1 < \cdots < k_n$ natural numbers, the following holds

$$\|\sum_{i=1}^{n} x_{k_i}\| \leqslant C$$

(ii) We say that $\{x_k\}_k$ generates ℓ_1 as a spreading model, if there exists a constant $c > 0$, such that for any $n \leqslant k_1 < \cdots < k_n$ natural numbers and $\lambda_1, \ldots, \lambda_n$ real numbers, the following holds

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An important property of $\mathfrak{X}_{\mathrm{ISP}}$

Proposition

Every seminormalized block sequence in $\mathfrak{X}_{ISP}$, has a subsequence generating either ℓ_1 or c_0 as a spreading model.

Properties of the space $\mathfrak{X}_{\text{ISP}}$

- The above proposition allows us to classify the weakly null sequences of $\mathfrak{X}_{\text{ISP}}$ as follows.

(i) The sequences of rank 0, i.e. the norm null sequences.

(ii) The sequences of rank 1, i.e. the sequences generating a c_0 spreading model.

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- The above proposition yields the following.

Corollary

Let Y be a closed and infinite dimensional subspace of $\mathfrak{X}_{ISP}$ and $S, T : Y \to Y$ strictly singular operators. Then the composition ST is a compact operator.

- Indeed, observe the following

(i) $T[B_Y]$ does not contain sequences of rank 2.

(ii) $ST[B_Y]$ does not contain sequences of rank 1 or 2, hence it is relatively compact.

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