Bourgain–Delbaen $L_\infty$-spaces, the Scalar-plus-Compact property and Invariant Subspaces

Spiros A. Argyros and Richard G. Haydon

Department of Mathematics, National Technical University of Athens
Mathematical Institute, University of Oxford

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Introduction

What we now call the “Bourgain–Delbaen construction” is a general method of constructing Banach spaces of the class $L_\infty$, inspired by a remarkable paper published in *Acta Math.* (1980).

By combining this technique with the machinery of hereditarily indecomposable constructions, as developed by Gowers, Maurey and Argyros, the speakers were able to exhibit in 2011 a separable Banach space $X_K$ on which every bounded linear operator has the form $\lambda I + K$ with $\lambda$ a scalar and $K$ a compact operator. The existence of such a space had been known as the Scalar-plus-Compact Problem. By the Aronszajn–Smith Theorem, the space $X_K$ also has the Invariant Subspace Property: that is to say every bounded linear operator on $X_K$ has a proper, non-trivial, closed invariant subspace. It is the first space known to have this property (though, of course, whether Hilbert space does is a famous open problem).
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In the last seven years, the BD method and variants of it have been used to construct a wide range of other examples: other workers involved include I. Gasparis, A. Manoussakis, M. Papadiamantis, A. Pelczar-Barwacz, P. Motakis, D. Puglisi, Th. Raikoftsalis, M. Świętek, M. Tarbard, A. Tolias, D. Zissimopoulou. We shall not have time to look in detail at all their contributions.
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The BD construction has also been useful in establishing new positive results (that is to say “not just counterexamples”), notably the embedding theorem of Freeman–Odell–Schlumprecht, and more recently in finding new proofs of some important theorems of Zippin. Again, time constraints will prevent us from looking at these developments in as much detail as we should wish.
The plan for the rest of the talk is as follows:

- $L_\infty$-spaces and the general BD-construction
- Standard BD-spaces
- The FOS embedding theorem
- Tsirelson spaces and “few-operators” constructions
- The space $\mathfrak{x}_K$ with very few operators
- The Invariant Subspace Property
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More precisely, a Banach space $X$ is a $L_\infty$-space if there is a constant $M$ such that, for every finite-dimensional subspace $E$ of $X$, there is a finite-dimensional subspace $F$ with $E \subseteq F \subseteq X$ that is $M$-isomorphic to a finite-dimensional $\ell_\infty$-space.
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So a separable space $X$ is $L_\infty,M$ iff there is an increasing sequence of finite-dimensional subspaces $E_1 \subseteq E_2 \subseteq \cdots$ such that $\bigcup_{n \in \mathbb{N}} E_n$ is dense in $X$ and $E_n$ is $M$-isomorphic to $\ell_\infty^{\dim E_n}$ for each $n$. 
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We are going to investigate a particular way to construct $L_\infty$-spaces, based on successive linear extensions.
We write $\mathbb{R}^\Gamma$ for the vector space of all real-valued functions on a set $\Gamma$, and when $F \subseteq \Gamma$ we identify $\mathbb{R}^F$ with the subspace of $\mathbb{R}^\Gamma$ consisting of functions that vanish off the subset $F$. 

We write $\mathbb{R}^\Gamma$ for the subspace of $\mathbb{R}^\Gamma$ consisting of functions of finite support. With the above convention, $\mathbb{R}^\Gamma = \bigcup_{F \subseteq \Gamma, \text{finite}} \mathbb{R}^F$. There is a natural duality between the spaces $\mathbb{R}^\Gamma$ and $\mathbb{R}^\Gamma^{\ast}$, given by $\langle f, x \rangle = \sum_{\gamma \in \Gamma} f(\gamma) x(\gamma)$, and generally we shall think of elements of $\mathbb{R}^\Gamma$ as vectors, while elements of $\mathbb{R}^\Gamma^{\ast}$ are seen as functionals.

For clarity (we hope!) functionals will be given names adorned with a star, so that the typical element of $\mathbb{R}^\Gamma^{\ast}$ will be denoted $f^{\ast}$, rather than $f$ as above. In particular, the function that takes the value 1 at $\gamma$, while vanishing elsewhere, will be denoted $e_\gamma$ if we think of it as the usual unit vector, and $e^{\ast}_\gamma$ if we think of it as the evaluation functional $x \mapsto x(\gamma)$. 

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Let \((\Gamma_n)_{n \in \mathbb{N}}\) be an increasing sequence of finite sets with union \(\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n\), and introduce the difference sets \(\Delta_1 = \Gamma_1\), \(\Delta_{n+1} = \Gamma_{n+1} \setminus \Gamma_n\). Suppose that, for each \(n\) and each \(\gamma \in \Delta_{n+1}\), we specify some \(c^*_{\gamma} \in \mathbb{R}^{\Gamma_n}\). For completeness, set \(c^*_{\gamma} = 0\) when \(\gamma \in \Gamma_1\).
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We can define, for each \(n\), a linear extension mapping \(i_{n+1,n} : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^{\Gamma_{n+1}}\), by

\[
i_{n+1,n}u(\gamma) = \begin{cases} 
  u(\gamma) & \text{if } \gamma \in \Gamma_n \\
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By forming compositions \(i_{m,m-1} \circ i_{m-1,m-2} \circ \cdots \circ i_{n+1,n}\) and letting \(m\) increase, we obtain, for each \(n\), a linear extension mapping \(i_n : \mathbb{R}^{\Gamma_n} \to \mathbb{R}^{\Gamma}\). Writing \(r_n\) for the restriction mapping \(\mathbb{R}^{\Gamma} \to \mathbb{R}^{\Gamma_n}\), we have the extension condition \(r_n i_n r_n = r_n\) and the compatibility condition \(i_m r_m i_n = i_n\) for \(n < m\).
Successive extension mappings (easy linear algebra)

Let \((\Gamma_n)_{n \in \mathbb{N}}\) be an increasing sequence of finite sets with union \(\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n\), and introduce the difference sets \(\Delta_1 = \Gamma_1\), \(\Delta_{n+1} = \Gamma_{n+1} \setminus \Gamma_n\). Suppose that, for each \(n\) and each \(\gamma \in \Delta_{n+1}\), we specify some \(c^*_\gamma \in \mathbb{R}^{\Gamma_n}\). For completeness, set \(c^*_\gamma = 0\) when \(\gamma \in \Gamma_1\).

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The images \(F_n = i_n[\mathbb{R}^{\Gamma_n}]\) form an increasing sequence of finite-dimensional subspaces of \(\mathbb{R}^\Gamma\). For each \(n\), \(P_n = i_n r_n\) is a projection on \(\mathbb{R}^\Gamma\) with image \(F_n\).
Continuing from the last slide, we introduce vectors $d_\gamma$ and functionals $d^*_\gamma$ defined as follows

\[
d_\gamma = i_n e_\gamma \quad \text{when } \gamma \in \Delta_n
\]
\[
d^*_\gamma = e^*_\gamma - c^*_\gamma
\]

Observation

1. $(d^*_\gamma)_{\gamma \in \Gamma}$ is an algebraic basis of $\mathbb{R}^{(\Gamma)}$.
2. $(d_\gamma)_{\gamma \in \Gamma}$ is the unique system in $\mathbb{R}^{\Gamma}$ that is biorthogonal to $(d^*_\gamma)_{\gamma \in \Gamma}$, and is an algebraic basis of $F = \bigcup_n F_n$.
3. The projection $P_n = i_n r_n : \mathbb{R}^{\Gamma} \to F_n$ is given by $P_n x = \sum_{\gamma \in \Gamma_n} \langle d^*_\gamma, x \rangle d_\gamma$.
4. The dual projection $P^*_n$ takes $\mathbb{R}^{(\Gamma)}$ onto $\mathbb{R}^{\Gamma_n}$ and is given by $P^*_n f^* = \sum_{\gamma \in \Gamma_n} \langle f^*, d_\gamma \rangle d^*_\gamma$. 

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Suppose that in the preceding discussion the extension operators \( i_n \) turn out to take values in \( \ell_\infty(\Gamma) \) with a uniform bound \( \|i_n\| \leq M \) for the norm of \( i_n \) as an operator from \( \ell_\infty(\Gamma_n) \) to \( \ell_\infty(\Gamma) \).
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Suppose that in the preceding discussion the extension operators $i_n$ turn out to take values in $l_\infty(\Gamma)$ with a uniform bound $\|i_n\| \leq M$ for the norm of $i_n$ as an operator from $l_\infty(\Gamma_n)$ to $l_\infty(\Gamma)$. A lot of good things happen.

- For each $n$, $F_n$ is $M$-isomorphic to $l_\infty(\Gamma_n)$, and hence the subspace $X$ of $l_\infty(\Gamma)$, defined as $X = \bigcup_{n \in \mathbb{N}} F_n$, is $L_{\infty,M}$.
- The functionals $d_\gamma^* = e_\gamma^* - c_\gamma^*$ form a Schauder basis of $l_1(\Gamma)$.
- The biorthogonal vectors $d_\gamma \in l_\infty(\Gamma)$ form a Schauder basis of $X$.
- The projections $P_n : l_\infty(\Gamma) \to F_n$ and $P_n^* : l_1(\Gamma) \to l_1(\Gamma_n)$ have norm at most $M$. 
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The problems of boundedness and duality

What we have just described is the General BD Construction and the resulting space $X$ is what we call a (general) BD-space. The set $\Gamma$, equipped with the compatible sequence of extension mappings $(i_n)$, or equivalently with the family of functionals $(c_\gamma^*)$, is what we call a BD-set.
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Although the use of extension operators might seem to limit the scope of constructions of this kind, Argyros, Gasparis and Motakis recently showed that every separable $\mathcal{L}_\infty$-space (and, in particular, every isomorphic predual of $\ell_1$) is isomorphic to a BD-space.
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What is missing at this level of generality is a useful condition on the functionals $c^*_\gamma$ that would guarantee uniform boundedness of the extensions $i_n$. It would also be nice to have a criterion for the canonically embedded isomorphic copy of $\ell_1(\Gamma)$ to make up the whole of $X^*$. 

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Bourgain–Delbaen $\mathcal{L}_\infty$-spaces
Plan

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- **Standard BD-spaces**
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  - Tsirelson spaces and “few-operators” constructions
  - The space $x_K$ with very few operators
  - The Invariant Subspace Property
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Standard BD-sets

The original paper of Bourgain and Delbaen gave a special way to define functionals $c^*_\gamma$ that guarantees the uniform boundedness of the extension operators $i_n$. We turn this into a definition.

We say that $\Gamma$ is a **Standard BD-set** with weight $\beta$ if, for all $n$ and all $\gamma \in \Delta_{n+1}$, the functional $c^*_\gamma$ has one of the forms

$$c^*_\gamma = \beta(I - P^*_s)b^*$$

with $0 \leq s \leq n$ and $b^* \in \text{ball } \ell_1(\Gamma_n \setminus \Gamma_s)$, or

$$c^*_\gamma = e^*_\xi + \beta(I - P^*_s)b^*$$

with $1 \leq s \leq n$, $\xi \in \Gamma_s$ and $b^* \in \text{ball } \ell_1(\Gamma_n \setminus \Gamma_s)$.
A crucial but easy estimate essentially due to B and D

Let $\Gamma$ be a standard BD set of weight $\beta$. Then the extension operators $i_n$ are uniformly norm bounded by $M = (1 - 2\beta)^{-1}$ and $X(\Gamma)$ is a $L_{\infty,M}$-space.
A crucial but easy estimate essentially due to B and D

Let $\Gamma$ be a standard BD set of weight $\beta$. Then the extension operators $i_n$ are uniformly norm bounded by $M = (1 - 2\beta)^{-1}$ and $X(\Gamma)$ is a $\mathcal{L}_\infty, M$-space.

We then call $X(\Gamma)$ a standard BD-space.
Recall that $c_\gamma^*$ is to be either $\beta(l - P_s^*)b^*$ or $e_\xi^* + \beta(l - P_s^*)b^*$. We call $\xi$ (when it exists) the *base* of $\gamma$, $b^*$ the *top* and $s$ the *cut*. 
Standard terminology for standard BD-spaces

Recall that $c_\gamma^*$ is to be either $\beta(I - P_s^*)b^*$ or $e_\xi^* + \beta(I - P_s^*)b^*$. We call $\xi$ (when it exists) the base of $\gamma$, $b^*$ the top and $s$ the cut.

So to define a standard BD-structure on a set $\Gamma$ we need first to express it as an increasing union of finite subsets $\Gamma_n$ and then to define functions

$$
\text{base} : \Gamma \to \Gamma \cup \{\text{undefined}\}, \quad \text{top} : \Gamma \to \text{ball } \ell_1(\Gamma), \quad \text{cut} : \Gamma \to \mathbb{N}
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satisfying the obvious constraints.
Recall that $c'_\gamma$ is to be either $\beta(I - P_s^*)b^*$ or $e_\xi^* + \beta(I - P_s^*)b^*$. We call $\xi$ (when it exists) the base of $\gamma$, $b^*$ the top and $s$ the cut.

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We still have freedom to specify the weight $\beta$ and it should be emphasized that different choices of $\beta$ can lead to Banach spaces of very different behaviour. There is even a degenerate case that is more useful than it might appear: we can set $\beta = 0$. 
A standard BD-set $\Gamma$ has a \textit{tree-structure} in which the immediate successors of $\xi$ are the elements $\gamma$ with base $\gamma = \xi$. This is intimately related to duality issues.
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**Theorem**

Let $\Gamma$ be a standard BD-set. Then the dual $X(\Gamma)^*$ is naturally identifiable with $\ell_1(\Gamma)$ if and only if there is no infinite sequence $(\xi_n)$ in $\Gamma$ with $\xi_n = \text{base } \xi_{n+1}$ for all $n$, that is to say, if and only if the tree structure on $\Gamma$ is well-founded.
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When $\Gamma$ is well-founded, it has a countable \textit{ordinal index} that becomes important when we look at the finer structure of $X(\Gamma)$. 
Plan

• $\mathcal{L}_\infty$-spaces and the general BD-construction
• Standard BD-spaces
• The FOS embedding theorem
• Tsirelson spaces and “few-operators” constructions
• The space $\mathcal{X}_K$ with very few operators
• The Invariant Subspace Property
It is an elementary fact that every separable Banach space is a quotient of $\ell_1$. The surprising theorem of FOS is that when $Y^*$ is a separable dual space then the quotient operator can itself be taken to be the dual of an embedding $Y \to X$ where $X$ is an isomorphic predual of $\ell_1$. 
The embedding theorem of Freeman, Odell and Schlumprecht

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Theorem [FOS (2011)]

Let $Y$ be a Banach space with separable dual $Y^*$. Then, for each $0 < \epsilon < \frac{1}{2}$, there exists a well-founded special BD-set $\Gamma$ and a $(1 + \epsilon)$-isomorphic embedding $Y \to X(\Gamma)$. 
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One approach to proving this theorem involves a careful construction of a certain $\epsilon$-net $\Gamma$ in ball $Y^*$, which we then equip with a standard BD structure in such that the natural map $J : Y \to \ell_\infty(\Gamma); (Jy)(\gamma) = \gamma(y)$ takes values in $X(\Gamma)$.
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He defined a norm on $\mathbb{R}(\mathbb{N})$ by

$$\|x\|_T = \sup_{f^* \in W} |\langle f^*, x \rangle|,$$

where the *norming set* $W \subset \mathbb{R}(\mathbb{N})$ is taken to be the smallest convex set such that $\pm e_n^* \in W$ for all $n$, and
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where the norming set $W \subset \mathbb{R}(\mathbb{N})$ is taken to be the smallest convex set such that $\pm e_n^* \in W$ for all $n$, and

\[ \frac{1}{2}(f_1^* + f_2^* + \cdots + f_n^*) \in W \quad \text{whenever} \quad f_1^*, \ldots, f_n^* \in W, \]
\[ \max \sup f_k^* < \min \sup f_{k+1}^* \quad (1 \leq k < n) \]
\[ \text{and} \quad n \leq \min \sup f_1^*. \]

The space $T$ is the completion of $\mathbb{R}(\mathbb{N})$ for this norm.
We can obviously generalize Tsirelson’s construction by replacing the constant $\frac{1}{2}$ with an arbitrary positive weight $\beta < 1$, and less obviously by replacing the condition $n \leq \min \supp f^*$ with $\{\min \supp f^*, \ldots, \min \supp f^n\} \in M$, for a suitable (regular) family $M$ of finite subsets of $\mathbb{N}$. We can even allow a sequence $(\beta_i)_{i \in \mathbb{N}}$ of distinct weights, with associated regular families $M_i$, obtaining a Mixed Tsirelson space.

Schlumprecht’s arbitrarily distortable space $S$ (1991) is a space of this type, and its discovery was a necessary step towards the celebrated counterexamples of Gowers (1993-4) and the Gowers–Maurey theory of hereditarily indecomposable (HI) Banach spaces.
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This is not the case for mixed Tsirelson spaces, though typically there is a class of special sequences (called RIS) for which $T_{\text{mixed}}$-upper estimates hold. This class is rich in the sense that every infinite-dimensional subspace contains a RIS.
A Banach space $X$ is said to have \textit{few operators} if every bounded linear operator $T$ on $X$ has the form $\lambda I + S$, with $S$ strictly singular (i.e. there is no infinite dimensional subspace on which $S$ is an isomorphism).
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A typical construction of a space with few operators would start with the ingredients $(\beta_i, M_i)_{i \in \mathbb{N}}$ of a mixed Tsirelson space, but add additional restrictions on the functionals $f_k^*$ that are allowed in the creation of new elements

$$\beta_i(f_1^* + f_2^* + \cdots + f_n^*)$$

of the norming set $W$. 

Using a coding method that goes back to Maurey and Rosenthal (1977) this can be done in such a way that for every operator $T$ on the resulting Banach space $X$ there is a scalar $\lambda$ such that $\|Tx_n - \lambda x_n\| \to 0$ for every RIS $(x_n)$. Since every infinite-dimensional subspace contains a RIS, $T - \lambda I$ is strictly singular.
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Spiros A. Argyros and Richard G. Haydon

Bourgain–Delbaen $L_\infty$-spaces
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We say that a Banach space has very few operators (or that $X$ has the Scalar plus Compact Property) if every $T$ can be written $T = \lambda I + K$ with $K$ compact.

Androulakis and Schlumprecht (2001) showed that the Gowers–Maurey space does not have very few operators by constructing an operator that is strictly singular and non-compact; the same seems to be true of other spaces constructed by mixed-Tsirelson techniques alone.

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Let $\Gamma$ be a well-founded standard BD set of weight $\beta$ and let $\gamma$ be an element of $\Gamma$. Recall that $c_\gamma^* = e_\xi^* + \beta(I - P_s^*)b^*$ if $\xi = \text{base } \gamma$ is defined.
The evaluation analysis

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Since $d^*_\gamma = e^*_\gamma - c^*_\gamma$ we can rewrite this as

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$$e_\gamma^* = \sum_{j=1}^{a} (\beta(I - P_{s_j}^*)b_j^* + d_{\xi_j}^*)$$  

where $\xi_a = \gamma$ and $\xi_j = \text{base } \xi_{j+1}$ for $j < a$. We call this identity the Evaluation Analysis of $\gamma$. 
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Bourgain–Delbaen $\mathcal{L}_\infty$-spaces
A basic inequality

In the evaluation analysis

\[ e_\gamma^* = \sum_{j=1}^{a} (\beta(I - P_{s_j}^*)b_j^* + d_{\xi_j}^*) \]

we call \( a \) the \textit{age} of \( \gamma \), and \( \{s_1, \ldots, s_a\} \) its \textit{history}.\]
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The presence of the sum \( \beta \sum_{j=1}^{a} ((I - P_{s_j}^*) b_j^* \) obviously puts us in mind of Tsirelson norms, and with effort we can prove the following.
A basic inequality

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**First Basic Inequality**

Let \( \Gamma \) be a well-founded, standard BD-set of weight \( \beta \). Let \( \mathcal{M} \) be a regular family that contains all histories of elements of \( \Gamma \). Then arbitrary normalized block sequences in \( X(\Gamma) \) admit \( T(\beta, \mathcal{M}) \)-upper estimates.
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$$e_\gamma^* = \sum_{j=1}^{a} (\beta (I - P_{s_j}^*) b_j^* + d_{\xi_j}^*)$$

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**First Basic Inequality**

Let $\Gamma$ be a well-founded, standard BD-set of weight $\beta$. Let $\mathcal{M}$ be a regular family that contains all histories of elements of $\Gamma$. Then arbitrary normalized block sequences in $X(\Gamma)$ admit $T(\beta, \mathcal{M})$-upper estimates.

I have cheated slightly: sometimes $\mathcal{M}$ needs to be *slightly* bigger than this.
The next step is to generalize our notion of standard BD-space to allow different elements to have different weights, drawn from a suitable sequence \((\beta_i)\) but subject to the condition weight \(\gamma = \text{weight base } \gamma\) when this makes sense.

We then take \(M_i\) to be a regular family that contains the histories of all \(\gamma\) of weight \(\beta_i\). We get a \(T\) mixed-upper estimate for a suitably defined class of sequences in \(X(\Gamma)\) (as always, referred to as RIS).

At this point, we do not seem to have advanced much from where we were before, since our upper estimates still apply only to some special sorts of sequences. But the local \(\ell_\infty\)-structure comes to the rescue, enabling us to prove a crucial lemma.

**Crucial Lemma**

Let \(\Gamma\) be a well-founded standard BD-space with mixed weights, let \(Z\) be any Banach space and let \(S: X(\Gamma) \to Z\) be a bounded linear operator. If \(\|S x_n\| \to 0\) for every RIS then \(\|S x_n\| \to 0\) for every bounded, weakly null sequence, and hence \(S\) is compact.
BD-sets with mixed weights

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To construct $\mathcal{X}_K$, we fix in advance a suitable fast-decreasing sequence of weights $\beta_i$ and a fast-increasing sequence of regular families $\mathcal{M}_i$. We then use recursion to build a standard BD-set $\Gamma^K$ with mixed weights, making sure as we add elements of $\Delta_{n+1}$ at stage $n+1$ that the history of any element of weight $\beta_i$ is in the family $\mathcal{M}_i$, but also using a coding to restrict the choice of top $\gamma$ when $\gamma$ has weight $\beta_i$ with $i$ odd.
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In a way that is not too different from earlier mixed-Tsirelson constructions, we show that the space $\mathcal{X}_K = X(\Gamma^K)$ is HI and that for every operator $T$ on $\mathcal{X}_K$ there is a scalar $\lambda$ such that $\|(T - \lambda I)x_n\| \to 0$ for every RIS.
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**Theorem**

The space $\mathcal{X}_K$ has the Scalar-plus-Compact Property.
The original motivation for the construction of the space $X_K$ was to find a hereditarily indecomposable space with dual isomorphic to $\ell_1$. The scalar-plus-compact property came as a (very agreeable!) surprise. It seemed natural to ask whether an isomorphic predual of $\ell_1$ with few operators automatically has very few. Tarbard answered this question in a striking way.

Theorem [Tarbard (2012)]

For each natural number $d$ there is a well-founded BD-set $\Gamma_d$ and a strictly singular operator $S$ on $X(\Gamma_d)$ such that $S_d + 1 = 0$ every bounded linear operator $T$ on $X(\Gamma_d)$ is uniquely expressible as $T = \lambda_0 I + d \sum_{r=1}^{d} \lambda_r S_r + K$, with $\lambda_0, \ldots, \lambda_d$ scalars and $K$ compact.
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Tarbard’s spaces

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**Theorem [Tarbard (2012)]**

For each natural number $d$ there is a well-founded BD-set $\Gamma^d$ and a strictly singular operator $S$ on $X(\Gamma^d)$ such that

- $S^{d+1} = 0$
- every bounded linear operator $T$ on $X(\Gamma^d)$ is uniquely expressible as

$$T = \lambda_0 I + \sum_{r=1}^{d} \lambda_r S^r + K,$$
We finish this part of the talk with two results that give partial answers to the question: “For which Banach spaces $Y$ does there exist an isomorphic embedding of $Y$ into a space with very few operators?”. It turns out that the answer is affirmative for some spaces that are in themselves not at all pathological.


Let $Y$ be a separable super-reflexive Banach space. Then $Y$ embeds in a space $X(\Gamma)$ with very few operators.

We note that this result applies in particular when $Y$ is a separable Hilbert space.

We conjecture that for spaces $Y$ with separable dual, the only obstruction to embeddability in a space with very few operators is the existence of a subspace of $Y$ isomorphic to $c_0$. While we have some improvements of the above theorem, we cannot prove this “best possible” result.
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**Theorem [S.A.–R.H.–Th. Raikoftsalis]**

There is a standard BD-set $\Gamma$ (not well-founded!) such that $X(\Gamma)$ contains $\ell_1$ and has very few operators.
Plan

- $L_\infty$-spaces and the general BD-construction
- Standard BD-spaces
- The FOS embedding theorem
- Tsirelson spaces and “few-operators” constructions
- The space $\mathcal{X}_K$ with very few operators
- The Invariant Subspace Property
The Invariant Subspace Problem

Definition

Let $X$ be a Banach space and $T$ a bounded linear operator on $X$. The operator $T$ admits a non-trivial invariant subspace if there exists a closed subspace $Y$ of $X$ with $Y \neq \{0\}$, $Y \neq X$ such that $T[Y] \hookrightarrow Y$. 

Spiros A. Argyros and Richard G. Haydon

Bourgain–Delbaen $L_\infty$-spaces
The Invariant Subspace Problem

**Problem**

Let $X$ be an infinite dimensional separable Banach space and $T : X \rightarrow X$ a bounded linear operator. Does $T$ admit a non trivial closed invariant subspace?

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Positive results

- In 1954 M. Aronszajn and K. T. Smith proved that every compact operator defined on an infinite dimensional Banach space admits a non trivial closed invariant subspace.

- In 1973 V. I. Lomonosov generalized the above result, by showing that in a complex Banach space $X$, every bounded linear operator $T \neq \lambda I$, for $\lambda \in \mathbb{C}$, which commutes with a non zero compact operator, admits a non trivial closed hyperinvariant subspace.

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P. Enflo and C. J. Read, in papers published in 1987 and 1984 respectively, present the first examples of operators on non reflexive, non classical Banach spaces, not admitting a non trivial closed invariant subspace.

Also, C. J. Read proved that any separable Banach space containing either $c_0$ or complemented $\ell_1$, admits an operator without a non trivial closed invariant subspace.

Since $c_0$ is separably injective, $c_0$ is a complemented subspace of $X$. The assumption that $\ell_1$ is a complemented subspace is necessary by our result with Raikoftsalis, mentioned earlier.

Moreover, in 1999 C. J. Read presents an example of a strictly singular operator on a non reflexive Banach space, which does not admit a non trivial closed invariant subspace.
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Moreover, in 1999 C. J. Read presents an example of a strictly singular operator on a non reflexive Banach space, which does not admit a non trivial closed invariant subspace.
A Banach space $X$ is said to have the Invariant Subspace Property (ISP), if every bounded linear operator on $X$ admits a non trivial closed invariant subspace. A Banach space satisfies the hereditary ISP if every closed subspace of it satisfies ISP.

Any Banach space with the scalar-plus-compact property, satisfies the Invariant Subspace Property.

The first example of a Banach space satisfying ISP is the $L_\infty$ space in the joint work with R. Haydon, mentioned before. However, it is not known if the space satisfies the hereditary ISP.
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The scalar-plus-compact problem for reflexive spaces remains open. Moreover, there is no known example of a bounded linear operator acting on a reflexive Banach space with no non trivial closed invariant subspace.

In the case of reflexive spaces every operator is weakly compact. We know from the classical Aronszajn - Smith theorem that compact operators admit non-trivial invariant subspaces. The question is if the weak compactness ensures the same property for weakly compact operators.
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Recently, in a joint work with Pavlos Motakis, we have constructed a reflexive Banach space with ISP. Furthermore, this is the first example of a Banach space, with hereditary ISP.
More precisely, a reflexive HI Banach space $\mathcal{X}_{ISP}$ is constructed satisfying the following.

(i) For every $Y$ closed subspace of $\mathcal{X}_{ISP}$, every $T \in \mathcal{L}(Y)$ is of the form $T = \lambda I + S$, with $S$ strictly singular.

(ii) For every $Y$ and $S$, $T : Y \to Y$ strictly singular operators, the composition $ST$ is a compact one.

(iii) For every infinite dimensional subspace $Y$ of $\mathcal{X}_{ISP}$, the space $S(Y)$ is non-separable.
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A reflexive space with ISP

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Every Banach space satisfying (i) and (ii), also satisfies the hereditary ISP.

Indeed, for every non zero strictly singular $S$, either $S^2 = 0$ and $\ker S$ is a non trivial closed hyperinvariant subspace for $S$, or $S$ commutes with $S^2$, which is non zero compact. In the latter case Lomonosov-Sirotkin Theorem yields that $S$ admits a non trivial closed hyperinvariant subspace.

The Calkin Algebra $\mathcal{C}(\mathcal{X}_{ISP}) := \mathcal{L}(\mathcal{X}_{ISP})/\mathcal{K}(\mathcal{X}_{ISP})$ is non-separable and for $S, T \in S(\mathcal{X}_{ISP})$, $[T][S] = 0$. This yields that $\mathcal{C}(\mathcal{X}_{ISP})$ admits at least two non equivalent complete multiplicative norms (Skillicorn).
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Let \( \{x_k\}_k \) be a Schauder basic sequence in a Banach space.

(i) We say that \( \{x_k\}_k \) generates \( c_0 \) as a spreading model, if there exists a constant \( C > 0 \), such that for any \( n \leq k_1 < \cdots < k_n \) natural numbers, the following holds

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\| \sum_{i=1}^{n} x_{k_i} \| \leq C
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(ii) We say that \( \{x_k\}_k \) generates \( \ell_1 \) as a spreading model, if there exists a constant \( c > 0 \), such that for any \( n \leq k_1 < \cdots < k_n \) natural numbers and \( \lambda_1, \ldots, \lambda_n \) real numbers, the following holds

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An important property of $\mathcal{X}_{ISP}$

**Proposition**

Every seminormalized block sequence in $\mathcal{X}_{ISP}$, has a subsequence generating either $\ell_1$ or $c_0$ as a spreading model.
The above proposition allows us to classify the weakly null sequences of $X_{ISP}$ as follows.

(i) The sequences of rank 0, i.e. the norm null sequences.

(ii) The sequences of rank 1, i.e. the sequences generating a $c_0$ spreading model.

(iii) The sequences of rank 2, i.e. the sequences generating an $\ell_1$ spreading model.
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Proposition

Let $Y$ be a closed and infinite dimensional subspace of $\mathcal{X}_{ISP}$, $S : Y \rightarrow Y$ a strictly singular operator and $\{x_k\}_k$ a seminormalized weakly null sequence in $Y$. Then $\{Sx_k\}_k$ is of rank strictly less than $\{x_k\}_k$.

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- The above proposition yields the following.
Corollary

Let $Y$ be a closed and infinite dimensional subspace of $\mathcal{X}_{ISP}$ and $S, T : Y \to Y$ strictly singular operators. Then the composition $ST$ is a compact operator. Indeed, observe the following

(i) $T[B_Y]$ does not contain sequences of rank 2.
(ii) $ST[B_Y]$ does not contain sequences of rank 1 or 2, hence it is relatively compact.
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(ii) $ST[B_Y]$ does not contain sequences of rank 1 or 2, hence it is relatively compact.
Corollary

Let $Y$ be a closed and infinite dimensional subspace of $X_{ISP}$ and $S, T : Y \to Y$ strictly singular operators. Then the composition $ST$ is a compact operator.

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(ii) $ST[B_Y]$ does not contain sequences of rank 1 or 2, hence it is relatively compact.