Constant Nonlocal Mean Curvature Surfaces

Mouhamed Moustapha Fall

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1 arxiv link: https://arxiv.org/abs/1804.04100
Soap films

Catenoid (Leonhard Euler, 1760)

Helicoid (Jean-Baptiste Marie Meusnier de La Place, 1776)

Scherk surface (Heinrich Scherk, 1834)
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Tools?

- Wire
- Soap
- Water
- Container, etc.
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Tools?

- Wire
- Soap
- Water
- Container, etc.
- Differential Equations
- Geometric measure theory
- Complex Analysis
- Differential Geometry, etc.
Area minimizing

They minimize the area functional,

These surfaces are called minimal surfaces,

They are solutions to the Plateau's problem: “to show the existence of a minimal surface with a given boundary”
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- They are solutions to the Plateau’s problem:
  "to show the existence of a minimal surface with a given boundary"
First variation of area

\[
\frac{\partial}{\partial t} \bigg|_{t=0} H_{\Sigma} - 1\left(\partial \Sigma_t\right) = \int_{\partial \Sigma} H_{\Sigma}(x) V \cdot V(x) \, d\sigma(x).
\]

The function \( H_{\Sigma} : \Sigma \to \mathbb{R} \) is called the Mean Curvature of the interface \( \partial E \).

1. Minimal hypersurfaces are critical points of the area functional thus \( H_{\Sigma}(x) = 0 \) for all \( x \in \Sigma \).
2. Constant Mean Curvature hypersurfaces are critical points of the area functional under volume constraint \( |\text{Int}(\Sigma_t)| = \text{Const} \).
First variation of area

Let $\Sigma$ be an orientable smooth surface with normal $\mathcal{N}$. Let $\Sigma_t = \{x + t\mathcal{N}(x), x \in \Sigma\}$. The function $H_{\Sigma}: \Sigma \to \mathbb{R}$ is called the Mean Curvature of the interface $\partial E$.

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First variation of area

Let $\Sigma$ be an orientable smooth surface with normal $V$. Let $\Sigma_t = \{x + tV(x), x \in \Sigma\}$.

First variation of area :

$$\left. \frac{\partial}{\partial t} \right|_{t=0} H^{N-1}(\partial \Sigma_t) = \int_{\partial \Sigma} H_{\Sigma}(x) \cdot V(x) \, d\sigma(x).$$
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Fractional area functional

\[ \int_{E} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N + \alpha}} \, dx \, dy. \]


Localized Fractional Perimeter of a set \( E \) in a set \( \Omega \)

\[ \int_{\mathbb{R}^2} \frac{1}{|1_{E}(x) - 1_{E}(y)|^{2\alpha}} \, dx \, dy. \]

Fractional Plateau’s problem (Caffarelli, Roquejoffre, Savin (2010)):

Fix a set \( E_0 \subset \mathbb{R}^N \), look for a minimizer of

\[ \inf \{ P_{\alpha, \Omega}(E) : E = E_0 \text{ on } \mathbb{R}^N \setminus \Omega \}. \]

Remark: As \( \alpha \to 1 \), fractional area \( P_{\alpha, \Omega}(E) \to H^{N - 1}(\partial E \cap \Omega) \) the area functional.

Davila, Caffarelli-Valdinoci, Ambrosio-De Philippis-Martinazzi.
Fractional area functional

- Fractional Perimeter of a bounded set

\[ P_\alpha(E) = (1-\alpha) \int_E \int_{\mathbb{R}^N \setminus E} \frac{1}{|x-y|^{N+\alpha}} \, dx \, dy \]
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- **Localized Fractional Perimeter of a set \( E \) in a set \( \Omega \)**
  \[ P_{\alpha,\Omega}(E) = \frac{1-\alpha}{2} \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|1_E(x) - 1_E(y)|^2}{|x-y|^{N+\alpha}} \, dx \, dy. \]
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Davilá, Caffarelli-Valdinoci, Ambrosio-De Phillipis-Martinazzi.
First variation of fractional area

Consider a smooth set $E$ and its local deformation $E_t = \{ x + tV(x), x \in E \}$. The function $H_\alpha \partial E : \partial E \to \mathbb{R}$ is called the nonlocal (or fractional) mean curvature.
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The function

\[
H_{\partial E}^\alpha : \partial E \rightarrow \mathbb{R}
\]

is called the nonlocal (or fractional) mean curvature.
Expression of Nonlocal Mean Curvature of hypersurfaces

Let $\Sigma \subset \mathbb{R}^N$ be a $C^2$ orientable hypersurface. The Mean Curvature of $\Sigma$ at a point $x \in \Sigma$ is given by

$$H_\Sigma(x) = \frac{1}{N-1} \text{div} V(x),$$

where $V$ is the (extended) unit normal vector field along $\Sigma$. The Nonlocal Mean Curvature (NMC) of $\Sigma$ at a point $x \in \Sigma$ is given by

$$H^{\alpha}_\Sigma(x) = \frac{2(1-\alpha)}{\alpha} \int_{\Sigma} (y-x) \cdot V(y) |y-x|^N + \alpha \, d\sigma(y),$$

for $\alpha \in (0,1)$. Caffarelli-Souganidis '08, Caffarelli-Roquejoffre-Savin '10.
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where $\mathcal{N}$ is the (extended) unit normal vector field along $\Sigma$.

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$$H_{\alpha \Sigma}(x) = 2(1 - \alpha) \int_{\Sigma} \frac{(y - x) \cdot \mathcal{N}(y)}{|y - x|^N} + \alpha \, d\sigma(y),$$

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Symplectic expression of (Nonlocal) Mean Curvature

Consider the signed indicator function of a set $E \subset \mathbb{R}^N$:

$$
\tau_E(x) := \begin{cases} 
-1 & \text{if } x \in E \\
+1 & \text{if } x \in E^c = \mathbb{R}^N \setminus E
\end{cases}
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The Mean Curvature at $x \in \partial E$ is

$$H_{\partial E}(x) := \frac{1}{2(N + 1)} \lim_{\varepsilon \to 0} \frac{-1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} \tau_E(y) \, dy.$$
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$$H^\alpha_{E}(x) := (1 - \alpha) \text{p.v.} \int_{\mathbb{R}^N} \frac{\tau_E(y)}{|x - y|^{N+\alpha}} \, dy.$$
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The Nonlocal Mean Curvature at \( x \in \partial E \) is

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H_E^\alpha(x) := (1 - \alpha) \text{ p.v.} \int_{\mathbb{R}^N} \frac{\tau_E(y)}{|x - y|^{N+\alpha}} \, dy = (1 - \alpha) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\tau_E(y)}{|x - y|^{N+\alpha}} \, dy.
\]

Remark: Up to a constant,

\[
H_{\partial E}(x) = \lim_{\alpha \to 1} H_E^\alpha(x_0).
\]
Some properties of the Nonlocal mean curvature

▶ Invariant by rigid motion:

\[ H_\partial E^\alpha = H_\partial \mathcal{R}(E) \quad \text{for } \mathcal{R} \text{ a rigid transformation} \]
Some properties of the Nonlocal mean curvature

- Invariant by rigid motion:
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- Scaling property:
  \[ H_{\partial E_r}^\alpha = r^{-\alpha} H_{\partial E}^\alpha \quad \text{with } E_r = rE \]
Some properties of the Nonlocal mean curvature

- Invariant by rigid motion:
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  \[ H_\partial^{\alpha E_r} = r^{-\alpha} H_\partial^\alpha E \quad \text{with } E_r = rE \]

- Comparison principle: If \( E_1 \subset E_2 \) with \( x \in \partial E_1 \cap \partial E_2 \), then \( H_\partial^{\alpha E_2} \leq H_\partial^{\alpha E_1} \)
Some properties of the Nonlocal mean curvature

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- Comparison principle: If \( E_1 \subset E_2 \) with \( x \in \partial E_1 \cap \partial E_2 \), then \( H_{\partial E_2}^\alpha \leq H_{\partial E_1}^\alpha \)

Remark: Local comparison does not hold i.e. \( E_1 \subset E_2 \) is necessary!
Some examples of (Nonlocal) minimal surfaces

Examples:

▶ The plane: The vector $x - y$ is always perpendicular to the normal of the plane!

▶ Helicoid: (Cinti, Davila, Del Pino, 2016) Other less trivial Nonlocal minimal surfaces?
Some examples of (Nonlocal) minimal surfaces

(Nonlocal) minimal surfaces $\Sigma$ are those satisfying

$$H_\Sigma(x) = H_\Sigma^\alpha(x) = \int_\Sigma \frac{(y - x) \cdot \mathcal{V}(y)}{|y - x|^{N+\alpha}} \, d\sigma(y) = 0$$

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Other less trivial Nonlocal minimal surfaces?
Some nontrivial Nonlocal minimal surfaces (Davilá, Del Pino, Wei, 2018)

For each of the following surfaces $\Sigma$ we have (provided $\alpha$ is close to 1)

$$H_\alpha^\Sigma(x) = \int_\Sigma (y - x) \cdot V(y) |y - x|^{N + \alpha} d\sigma(y) = 0.$$
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- The nonlocal Catenoid
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Trivial examples of Constant (Nonzero) Nonlocal Mean Curvature surfaces

Curves and surfaces $\Sigma$ with $H^{\alpha}_{\Sigma}(x) = \int_{\Sigma} \left( y - x \right) \cdot V(y) \left| y - x \right| N + \alpha \, d\sigma(y) = \text{Const}, \quad \alpha \neq 0$.

Of course parallel lines/planes have zero classical mean curvature!
Trivial examples of Constant (Nonzero) Nonlocal Mean Curvature surfaces

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Of course parallel lines/planes have zero classical mean curvature!
NMC of parallel straight lines

Let \( \Sigma \) be the nonlocal mean curvature at \( x = (x_1, 0) \in L_1 \), integrate over \( L_2 \):

\[
H_\alpha \Sigma(x) = (1 - \alpha) \int \Sigma(y - x) \cdot V(y) |y - x|^{N+\alpha} \, d\sigma(y)
\]
The NMC of parallel straight lines is the interaction of a point in "Line 1" with the whole "Line 2".

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\[ = (1 - \alpha) \int_{\mathbb{R}} \frac{1}{(1 + t^2)^{(N+\alpha)/2}} dt. \]
Theorem (Alexandrov (1956)) The unit sphere is the only bounded connected surface with nonzero Constant Mean Curvature.

Theorem (Cabrè, Solà-Morales, F., Weth / Ciraolo, Figalli, Maggi and Novaga, '15) The sphere is the only bounded (not necessarily connected) nonzero Constant Nonlocal Mean Curvature hypersurfaces.

Idea of proof: Alexandrov's moving plane argument...
Rigidity I. Characterizations of the sphere

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**Idea of proof:** Alexandrov’s moving plane argument...
The moving plane method

Step 1: Consider a smooth set $E$ and call $E_\lambda$ its reflection with respect to the hyperplane $\{x_1 = \lambda\}$.

Step 1: Then move the reflected set $E_\lambda$ toward $E$ until:

- Interior touching at $x \in \partial E \cap \partial E_\lambda$
- Non-transversal intersection $e_1 \in T_x \partial E = T_x \partial E_\lambda$
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![Diagram showing the moving plane method](image)
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Comparison 1: $H_E^\alpha(x) := p.v. \int_{\mathbb{R}^N} \frac{\tau_E(y)}{|x-y|^{N+\alpha}} \, dy$

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Because $E$ and $E_\lambda$ have the same NMC,

$$H_{\partial E}^\alpha(x) - H_{\partial E_\lambda}^\alpha(x) = 0 \implies E = E_\lambda.$$

Non-transversal intersection $e_1 \in T_x \partial E = T_x \partial E_\lambda$
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Because $E$ and $E_\lambda$ have both Constant NMC,

$$\partial e_1 H^\alpha_{\partial E}(x) - \partial e_1 H^\alpha_{\partial E_\lambda}(x) = 0 \quad \implies \quad E = E_\lambda.$$
Classification of graphs with Constant (Nonlocal) Mean Curvature?
Nonlocal Mean Curvature acting on graphs

Let $u: \mathbb{R}^{N-1} \to \mathbb{R}$ and the subgraph $\mathcal{E}_u = \{ (x, t) \in \mathbb{R}^{N-1} \times \mathbb{R} : t < u(x) \}$.

The mean curvature $H_{\partial \mathcal{E}_u}(x) = -\text{div} \nabla u \sqrt{1 + |\nabla u|^2}(x)$

The nonlocal mean curvature $H_\alpha_{\partial \mathcal{E}_u}(x) = \int_{\mathbb{R}^{N-1}} u(x) - u(y) \frac{1}{|x-y|^{N+\alpha}} dy,$ where $Q(r) = (1 - \alpha) \int_{1-\alpha}^1 \frac{d\tau}{(1+\tau^2 r^2)^{(N+\alpha)/2}}$.

“A quasilinear nonlocal operator of order $1 + \alpha$”

Remark: For $\alpha \in (0, 2], \sup_{\mathbb{R}^{N-1}} |\nabla u| \ll 1 \Rightarrow H_\alpha_{\partial \mathcal{E}_u} \approx (-\Delta)^{(1+\alpha)/2} u$.
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Let \( u : \mathbb{R}^{N-1} \rightarrow \mathbb{R} \) and the subgraph

\[ E_u = \{(x, t) \in \mathbb{R}^{N-1} \times \mathbb{R} : t < u(x)\}. \]
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$$H_{\partial E_u}(x) = -\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)(x)$$

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\[
H_{\partial E_u}^\alpha(x) = \int_{\mathbb{R}^{N-1}} \frac{u(x) - u(y)}{|x - y|^{N+\alpha}} Q \left( \frac{u(x) - u(y)}{|x - y|} \right) dy,
\]

where \( Q(r) = (1 - \alpha) \int_{-1}^{1} \frac{d\tau}{(1 + \tau^2 r^2)^{\frac{N+\alpha}{2}}} \).
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**Remark:** For $\alpha \in (0, 2]$,

$$\sup_{\mathbb{R}^{N-1}} |\nabla u| \ll 1 \implies H_{\partial E_u}^\alpha \simeq (-\Delta)^{\frac{1+\alpha}{2}} u$$
Rigidity II: Bernstein’s Problem

Bernstein's Problem: Is there a (nonlocal) minimal graph in $\mathbb{R}^N$ that is not a hyperplane?

In the classical case:


In the nonlocal case:

- No, if $N \leq 2$. Figalli, Valdinoci (2017).
- Open, if $N \geq 3$. A challenging open problem...!
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Delaunay (1841): Surfaces of revolution in $\mathbb{R}^3$ with constant, nonzero mean curvature are surfaces of revolution of roulettes of the conics. They are cylinders, unduloids, spheres, and nodoids. Rolling an ellipsoid and tracing the focus then rotate the resulting curve around the $z$-axis. What about the nonlocal case?
Rigidity III: Rotationally symmetric Constant Mean Curvature surfaces

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Unduloids

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Curves and Surfaces with Constant Nonlocal Mean Curvature

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\{ (x, z) \in \mathbb{R}^m \times \mathbb{R}^n : |z| = \phi(x) \}
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- \( n=1, m=2 \) by Niang, Minlend and Thiam (Preprint 2018)
Global bifurcation from cylinder to spheres with CNMC (Open problem!)  

Local bifurcation: Unduloid varies from cylinder to a string-of-pearl. Pearls are disjoint round-spheres.

In the non-local bifurcation: Unduloid varies from cylinder to a string-of-pearl. Pearls are disjoint not-round-spheres.

Hints: Dàvila, Del Pino, Dipierro, Valdinoci (2016) and Cabré, F., Weth (2017)
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Consider the hypersurface $\Sigma_r$ made of periodic array of round spheres:

$$\Sigma_r := S^{N-1} + r\mathbb{Z}$$
Near spheres with CNMC

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- It has constant classical mean curvature.
- It has non-constant nonlocal mean curvature. However, Theorem (Cabr´ e - F. - Weth, 2018) The hypersurfaces $\Sigma_r$ can be perturbed (for large $r$) to a periodic hypersurface with Constant Nonlocal Mean Curvature.

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**Theorem (Cabré - F. - Weth, 2018)** The hypersurfaces $\Sigma_r$ can be perturbed (for large $r$) to a periodic hypersurface with Constant Nonlocal Mean Curvature. More precisely, they have the form:
Mean curvature flow approximations by Lévy diffusion heat flow

Let $E$ be a smooth open set. Consider $X_{\alpha t}$ an $\alpha$-stable Lévy process, (with $X_{2t} = B_t$ a Brownian motion).

Solution to the nonlocal heat diffusion

$$u_t(x) = \mathbb{E}_x(\tau_{E}(X_{\alpha t})),$$
$$u_0 = \tau_{E} = 1_{\mathbb{R}^N \setminus E}^{-1}.$$

For small times $t_0 > 0$, $\bigcup_{t < t_0} \{u_t = 0\} \sim \begin{cases} 
\text{Mean Curvature motion of } \partial E 
\text{if } \alpha \in (1, 2] 
\text{Nonlocal Mean Curvature motion of } \partial E 
\text{if } \alpha \in (0, 1) 
\end{cases}$

▶ For $\alpha = 2$, Merriman-Bence-Osher (1992), Evans (1993),
▶ For $\alpha \in (0, 2)$, Caffarelli-Souganidis (2008).
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$$ \bigcup_{t < t_0} \{ u_t = 0 \} \sim \begin{cases} 
\text{Mean Curvature motion of } \partial E & \text{if } \alpha \in (1, 2] \\
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- For $\alpha = 2$, Merriman-Bence-Osher (1992), Evans (1993),
Mean curvature flow approximations by Lévy diffusion heat flow

Let $E$ be a smooth open set.

Consider $X_t^\alpha$ an $\alpha$-stable Lévy process,
(with $X_t^2 = B_t$ a Brownian motion).

Solution to the nonlocal heat diffusion

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Remarks and Open problems

The set of CNMC surfaces found so far captures the geometry and distributions of periodic patterns in some block copolymer phase diagram.

Investigate the nonlocal counterpart of the Gyroid.

The existence of a nonlocal catenoid in the full range of the fractional parameter $\alpha$?

Existence of nonlocal minimal surfaces with genus e.g. Costa's surface

Existence of periodic nonlocal minimal surface e.g. Schwartz minimal surface and Schreck surfaces.

Connection of the theory of CNMC to Overdetermined problems!
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  Connection of the theory of CNMC to Overdetermined problems!? 
An elementary problem in fluid dynamics (Serrin 1971)

Consider a viscous incompressible laminar fluid flow in a long cylinder with given cross section \( \Omega \subset \mathbb{R}^2 \).

Stationary velocity field \( \mathbf{v} = (0, 0, u) \) with \( u : \Omega \to \mathbb{R} \).

Then \( u \) solves the Poisson problem

\[
-\Delta u = \delta \ell \eta \quad \text{in } \Omega,
\]

\( u = 0 \) on \( \partial \Omega \) (no slip boundary condition).

Here:

\( \delta \sim \text{pressure difference} \),

\( \ell \sim \text{cylinder length} \),

\( \eta \sim \text{viscosity} \).

Question: Which domains \( \Omega \) give rise to a constant tangential stress?

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Theorem (Serrin 1971 and Weinberger 1971)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^2$-boundary such that the overdetermined problem

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admits a positive solution.

Then $\Omega = B_r(0)$ and $u(x) = \frac{1}{2^N}(r^2 - |x|^2)$.

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Conjecture (Berestycki, Caffarelli, Nirenberg 1997)

Let $\Omega \subset \mathbb{R}^N$ be an unbounded sufficiently regular domain such that $\mathbb{R}^N \setminus \overline{\Omega}$ is connected, and let $f \in C^1(\mathbb{R})$. If the overdetermined problem

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Then

- $\Omega$ is an affine half space, or
- $\Omega = B^c$ for a ball $B \subset \mathbb{R}^N$, or
- $\Omega$ is a product of the form $\mathbb{R}^j \times B$ bzw. $\mathbb{R}^j \times B^c$ (modulo $O(N)$)

The conjecture is false!

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- del Pino-Pacard-Wei (2015), monostable nonlinearity $f$.
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How about the original torsion problem: $f \equiv 1$?
Non-trivial solutions\footnote{M.M. F., I.A. Minlend and T. Weth, (2016)}
Non-trivial solutions \(^3\)

Serrin domains of the form \(\Omega_\varphi := \{(x, z) \in \mathbb{R}^m \times \mathbb{R}^n : |z| < \varphi(x)\}\), i.e.

\[^3\text{M.M. F., I.A. Minlend and T. Weth, (2016)}\]
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\[\text{Diagram:} \quad n=1, m=1\]
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\[n=1, m=1\]

\[n=1, m=2\]

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Thank you for your attention!