Decouplings and applications

Ciprian Demeter, IU Bloomington

ICM, August 2018
Sketch of the talk

1. Some motivation: exponential sums
2. What is a Fourier analytic decoupling?
3. Decouplings imply moment estimates for exponential sums
4. Examples
Let Λ be a collection of frequency points ξ on some **curved**, compact manifold S of diameter \( \sim 1 \) in \( \mathbb{R}^n \). Examples include hypersurfaces, such as the truncated paraboloid

\[
\mathbb{P}^{n-1} = \{ \xi = (\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) : |\xi_i| \leq 1 \}
\]

and the moment curve

\[
\Gamma_n = \{ (t, t^2, \ldots, t^n) : t \in [0, 1] \}.
\]
Let $B_R$ be a ball/cube with diameter $R \geq 1$. Let also $a_\xi \in \mathbb{C}$.

**Notation:** $e(z) = e^{2\pi iz}$.

**Question**

*How to estimate the $L^p$ moment of the exponential sum, i.e.*

$$\| \sum_{\xi \in \Lambda} a_\xi e(x \cdot \xi) \|_{L^p(B_R)}$$

**Figure:** Level curves for $e(x \cdot \xi_1)$ and $e(x \cdot \xi_2)$ (here $\xi_1, \xi_2 \in \mathbb{R}^2$)
Let us assume the points in $\Lambda$ are $\delta$-separated for some $\delta > 0$, but not necessarily inside a lattice. We will be interested in the smallest radius $R$ that captures well enough the almost periodicity of the exponential sum.

A simple computation in the lattice case suggests that this is $R \sim \delta^{-2}$ for hypersurfaces in $\mathbb{R}^n$ and $R \sim \delta^{-n}$ for curves in $\mathbb{R}^n$. One example that captures both cases is the parabola $\mathbb{P}^1$.

Changing variables and using periodicity (here $\delta = \frac{1}{N}$)

$$\left\| \sum_{k=1}^{N} a_k e(x \cdot \left( \frac{k}{N}, \frac{k^2}{N^2} \right)) \right\|_{L^p([0,N^2] \times [0,N^2],dx)} =$$

$$N^4_p \left\| \sum_{k=1}^{N} a_k e(kx_1 + k^2x_2) \right\|_{L^p([0,1]^2, dx_1 dx_2)}.$$
Motivation: This example generalizes to higher dimensions in two ways, motivated by questions from PDEs and Number Theory

1. On $\mathbb{P}^{n-1}$. Strichartz estimates for Schrödinger equation on tori initiated by Bourgain (1993) amount to estimating

$$\left\| \sum_{k=(k_1,\ldots,k_{n-1}) \atop 1 \leq k_i \leq N} a_k e(x \cdot \left( \frac{k_1}{N}, \ldots, \frac{k_{n-1}}{N}, \frac{k_1^2 + \cdots + k_{n-1}^2}{N^2} \right)) \right\|_{L^p([0,N^2]^n,dx)}$$

2. On $\Gamma_n$. The Main Conjecture in the Vinogradov Mean Value Theorem amounts to estimating

$$\left\| \sum_{k=1}^{N} e(x \cdot \left( \frac{k}{N}, \frac{k^2}{N^2}, \ldots, \frac{k^n}{N^n} \right)) \right\|_{L^p([0,N^n]^n,dx)}$$
In this type of problems there is always a range \( 2 \leq p \leq p_c \) (the **critical exponent** \( p_c \) will depend on the manifold) such that the following square root cancellation occurs (\( \| a_\xi \|_{L^2} = (\sum_\xi |a_\xi|^2)^{1/2} \))

\[
R^{-\epsilon} \| a_\xi \|_{L^2} \lesssim_\epsilon \left( \frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(x \cdot \xi) \right|^p \ dx \right)^{1/p} \lesssim_\epsilon R^\epsilon \| a_\xi \|_{L^2}
\]

for each \( \epsilon > 0 \). This can be viewed a (weak, average) form of almost periodicity and also as a reverse Hölder’s inequality.

This estimate is an easy consequence of \( L^2 \) almost orthogonality when \( p = 2 \), and its gets harder to prove as \( p \) gets closer to \( p_c \).

We will investigate these questions about exponential sums in the more general framework of **decouplings**.
Let \((f_j)_{j=1}^N\) be \(N\) elements in a normed space \((X, \| \cdot \|_X)\). In this generality the triangle inequality

\[
\| \sum_{j=1}^N f_j \|_X \leq \sum_{j=1}^N \| f_j \|_X
\]

is the best estimate available for the norm of the sum of \(f_j\). When combined with the Cauchy–Schwarz inequality, it leads to

\[
\| \sum_{j=1}^N f_j \|_X \leq N^{1/2} \left( \sum_{j=1}^N \| f_j \|_X^2 \right)^{1/2}.
\]

Choosing \(X = L^1(\mathbb{R}^n)\) and positive functions \(f_j\) with equal \(L^1\) norms shows that the inequality from above can be sharp.
However, if $X$ is a Hilbert space and if $f_j$ are pairwise orthogonal then we have a stronger inequality (in fact an equality), a abstract form of square root cancellation

$$\left\| \sum_{j=1}^{N} f_j \right\|_X \leq \left( \sum_{j=1}^{N} \| f_j \|_X^2 \right)^{1/2}.$$

An example which is ubiquitous in Fourier analysis is when $X = L^2(\mathbb{R}^n)$, and the $f_j$ are functions whose Fourier transforms are disjointly supported.

It is natural to ask whether there is an analogous phenomenon in $L^p(\mathbb{R}^n)$ when $p \neq 2$, in the absence of Hilbert space orthogonality.
Given a smooth function $\psi : U \rightarrow \mathbb{R}^{n-d}$ we define the $d$-dimensional manifold in $\mathbb{R}^n$

$$\mathcal{M} = \mathcal{M}^\psi = \{ (\xi, \psi(\xi)) : \xi \in U \}.$$ 

Examples of hypersurfaces ($d = n - 1$) include the paraboloid

$$\mathbb{P}^{n-1} = \{ (\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \ldots + \xi_{n-1}^2) : |\xi_i| < 1 \},$$

the hemispheres

$$\mathbb{S}^{n-1}_{\pm} = \{ (\xi, \pm \sqrt{1 - |\xi|^2}) : |\xi| < 1 \},$$

the truncated cone

$$\mathbb{C}one^{n-1} = \{ (\xi, |\xi|) : 1 < |\xi| < 2 \}.$$ 

A "nice" curve ($d = n - 1$) is the moment curve

$$\Gamma_n = \{ (t, t^2, \ldots, t^n) : t \in [0, 1] \}.$$
Let $\Theta_M(\delta)$ be a partition of $M$ into sets $\theta$ of “size” $\sim \delta$. For each $\theta$ let $\mathcal{P}_\theta F$ be the Fourier restriction of $F$ to $\theta$

$$\mathcal{P}_\theta F = (\hat{F}1_\theta)^\vee \text{ or } \mathcal{P}_\theta F = \hat{F}1_\theta.$$

If $\hat{F}$ is supported on $M$ then

$$F = \sum_{\theta \in \Theta_M(\delta)} \mathcal{P}_\theta F$$

**Problem (Decoupling for manifolds)**

*Find the range $2 \leq p \leq p_c$ such that the following $l^2$ decoupling holds for each $F$ with Fourier transform supported on $M$*

$$\|F\|_{L^p(B(0,R))} \lesssim_\varepsilon R^\varepsilon \left( \sum_{\theta \in \Theta_M(\delta)} \|\mathcal{P}_\theta F\|_{L^p(B(0,R))}^2 \right)^{1/2}.$$
The fact that decouplings imply exponential sum estimates relies on the fact that for each $\xi \in \mathbb{R}^n$ the Fourier transform of the Dirac delta distribution

$$\delta_\xi(\eta) := \begin{cases} 1, & \eta = \xi \\ 0, & \eta \neq \xi \end{cases}$$

is the exponential

$$\hat{\delta}_\xi(x) = e(x \cdot \xi).$$

**Figure:** Partition of $\mathcal{M}$ and points $\xi_\theta$
Bourgain’s observation (2013): To get from...

**Theorem (Decoupling for $\mathcal{M}$)**

$$\|F\|_{L^p(B(0,R))} \lesssim \varepsilon R^\varepsilon \left( \sum_{\theta \in \Theta_{\mathcal{M}}(\delta)} \|\mathcal{P}_\theta F\|_{L^p(B(0,R))}^2 \right)^{1/2}.$$  

...to the exponential sum estimate (reverse Hölder)

**Theorem (Exponential sum estimate for $\mathcal{M}$)**

For each $\theta \in \Theta_{\mathcal{M}}$ let $\xi_\theta \in \theta$ and $a_\theta \in \mathbb{C}$. Then

$$|B(0, R)|^{-1/p} \left\| \sum_{\theta \in \Theta_{\mathcal{M}}(\delta)} a_\theta e(\xi_\theta \cdot x) \right\|_{L^p(B(0,R))} \lesssim \varepsilon R^\varepsilon \left( \sum_{\theta} |a_\theta|^2 \right)^{1/2}$$

simply use (a smooth approximation of)

$$\hat{F}(\xi) = \sum_{\theta \in \Theta_{\mathcal{M}}(\delta)} a_\theta \delta_{\xi_\theta}$$
The first decoupling was proved by Wolff in 2000 for the cone in $\mathbb{R}^3$. His range of $L^p$ estimates was incomplete, but good enough to have very interesting consequences (such as local smoothing for the wave equation in some range).

His proof connected decoupling (a highly oscillatory problem) with incidence geometry problems (counting circle tangencies).
The next big breakthrough came in 2013 (Bourgain). He proved a decoupling for the paraboloid $\mathbb{P}^{n-1}$ in the (incomplete) range $2 \leq p \leq \frac{2n}{n-1}$. Bourgain’s new tool is the multilinear Kakeya inequality.
Consider $n$ families $\mathcal{T}_j$ consisting of $R \times R^{1/2} \times \ldots \times R^{1/2}$ tubes $T \subset B_{4R}$ in $\mathbb{R}^n$ having the following property

**Transversality:** The direction of the long axis of $T \in \mathcal{T}_j$ is in a small neighborhood of $e_j = (0, \ldots, 1, \ldots, 0)$

Then we have the following inequality ($\mathcal{I}$ is the average integral)

$$\mathcal{I} \left( \prod_{j=1}^{n} F_j \right)^{\frac{1}{n}} \lesssim_{\epsilon R^\epsilon} \left[ \prod_{j=1}^{n} \int_{B_{4R}} |F_j|^{\frac{1}{n}} \right]^{\frac{n}{n-1}}$$

(1)

for all functions $F_j$ of the form

$$F_j = \sum_{T \in \mathcal{T}_j} c_T 1_T, \quad c_T \geq 0.$$
In 2014 Bourgain and I extended the decoupling for the paraboloid to the full range \(2 \leq p \leq p_c = \frac{2(n+1)}{n-1}\). We combine

- wave packet decompositions and the multilinear Kakeya inequality
- \(L^2\) orthogonality
- the equivalence between multilinear and linear decoupling
- a bootstrapping argument that makes use of many scales

Also, we used decoupling for the paraboloid as a black box to produce a fairly short argument for the decoupling for the cone in \(\mathbb{R}^n\) in the full expected range. Our argument avoids the use of circle tangencies.
In 2015, Guth Bourgain and I have also completed the decoupling theory for the moment curve. Our argument uses the full range of multilinear Kakeya estimates for shapes ranging between “long tubes” and “thin plates”.

As an immediate corollary we proved the Main Conjecture in the Vinogradov Mean Value Theorem for all $n$. 
For each integers $s \geq 1$ and $n, N \geq 2$ denote by $J_{s,n}(N)$ the number of integral solutions for the following system

$$X_1^i + \ldots + X_s^i = Y_1^i + \ldots + Y_s^i, \quad 1 \leq i \leq n,$$

with $1 \leq X_1, \ldots, X_s, Y_1, \ldots, Y_s \leq N$.

Note the lower bound $N^s \lesssim J_{s,n}(N)$ (counting trivial solutions)

---

**Theorem (Main Conjecture in Vinogradov’s Mean Value “Theorem”)**

*For each $\epsilon > 0$ and $n, N \geq 2$ we have the upper bound*

$$J_{\frac{n(n+1)}{2},n}(N) \lesssim \epsilon N^{\frac{n(n+1)}{2}} + \epsilon.$$
Example: \( n = 2, s = 3 \) then \( J_{3,2}(N) \lesssim_{ \varepsilon } N^{3+\varepsilon} \) (easy, folklore)

\[
\begin{align*}
X_1 + X_2 + X_3 &= Y_1 + Y_2 + Y_3 \\
X_1^2 + X_2^2 + X_3^2 &= Y_1^2 + Y_2^2 + Y_3^2
\end{align*}
\]

Example: \( n = 3, s = 6 \) then \( J_{6,3}(N) \lesssim_{ \varepsilon } N^{6+\varepsilon} \) (Wooley 2014, through efficient congruencing)

\[
\begin{align*}
X_1 + \ldots + X_6 &= Y_1 + \ldots + Y_6 \\
X_1^2 + \ldots + X_6^2 &= Y_1^2 + \ldots + Y_6^2 \\
X_1^3 + \ldots + X_6^3 &= Y_1^3 + \ldots + Y_6^3
\end{align*}
\]
Theorem (Bourgain, D., Guth 2016, Wooley 2017)

The upper bound

\[ J_{\frac{n(n+1)}{2},n}(N) \lesssim \epsilon N^{\frac{n(n+1)}{2}} + \epsilon \]

holds for all \( n \geq 2 \).

Our proof relies on the fact that the number \( J_{s,n}(N) \) has the following analytic representation

\[ J_{s,n}(N) = \int_{[0,1]^n} | \sum_{j=1}^N e(x_1 j + x_2 j^2 + \ldots + x_n j^n) |^{2s} \, dx_1 \ldots dx_n. \]

The relevant exponential sum estimate is an immediate consequence of a decoupling for the moment curve.
The proofs of our decoupling theorems rely solely on Fourier analysis and incidence geometry (multilinear Kakeya estimates), not on number theory.

In the problems that can be tackled with decoupling tools, the rigid algebraic structure of integers is irrelevant, it is enough to have separation. In light of this, it seems reasonable to conclude that such results as the Vinogradov Mean Value Theorem belong to Analysis, not Number Theory.

Our technology has applications outside the realm of exponential sums: we can decouple whole arcs, not just point masses.
Some examples of manifolds with sharp decoupling theory

- Hypersurfaces in $\mathbb{R}^n$ with nonzero Gaussian curvature. **Many applications:** Optimal Strichartz estimates for Shrödinger equation on both rational and irrational tori in all dimensions, improved $L^p$ estimates for the eigenfunctions of the Laplacian on the torus, etc.

- The cone (zero Gaussian curvature) in $\mathbb{R}^n$. **Many applications:** progress on Sogge’s “local smoothing conjecture for the wave equation”, Bergman projections, Gauss circle problem.

- Curves with torsion in $\mathbb{R}^n$. **Application:** Vinogradov’s Mean Value Theorem.
More exotic manifolds

- Two dimensional surfaces in $\mathbb{R}^4$ **Application:** Bourgain used this to improve the estimate in the Lindelöf hypothesis for the growth of Riemann zeta

- Surfaces in $\mathbb{R}^9$ and beyond. **Application:** All Parsell-Vinogradov systems (Bourgain, D.-Guo. and Guo-Zhang)
Open problems

Consider the generalized additive energy

$$E_n(A) = |\{(a_1, \ldots, a_{2n}) \in A^{2n} : a_1 + \ldots + a_n = a_{n+1} + \ldots a_{2n}\}|$$

1. Prove (or disprove) that $E_2(A) \lesssim \epsilon |A|^{2+\epsilon}$ if $A \subset S^2$. Known for subsets of the paraboloid $A \subset P^2$

2. Prove (or disprove) that $E_3(A) \lesssim \epsilon |A|^{3+\epsilon}$ if $A \subset S^1$ or $A \subset P^1$ For $S^1$, this follows from the unit distance conjecture. Best known unconditional bound (Bombieri-Bourgain) is $|A|^{7/2}$ via the Szemerédi-Trotter theorem.

All these estimates follow from our decoupling theorems in the case of well-separated points.