On the Dyadic Hilbert Transform

Topics in Weights and Multi-parameter Analysis

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The Hilbert transform...

...on the real line

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If f is sufficiently integrable then H gives access to its harmonic conjugate function.

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In this talk we present different ideas that shed new light on how we view classical singular operators.

The Hilbert transform

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We will see that it is also intimately tied to a wavelet system.

The dyadic Hilbert transform

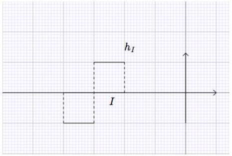
Let

$$\mathcal{D} = \{2^n[0,1[+ k2^n : k, n \in \mathbb{Z}\}$$

be the standard dyadic grid with reference point 0 and

$$h_I = \frac{1}{\sqrt{|I|}}(\chi_{I_+} - \chi_{I_-})$$

the Haar wavelet associated to the dyadic interval I.



The dyadic Hilbert transform

It turns out that the Hilbert transform is a multiple of an average of the dyadic **Shift operator**

$$h_I \mapsto h_{I_-} - h_{I_+}$$

when considering translates and dilates of the standard dyadic grid

$$\mathcal{D}^{\alpha,r} = \{r2^n[\alpha, \alpha+1[+ kr2^n : k, n \in \mathbb{Z}\}$$

The dyadic Hilbert transform

Denoting the representing kernel of the dyadic Shift operator based on the grid $\mathcal{D}^{\alpha,r}$ as $K^{\alpha,r}$ there holds

Theorem (P)

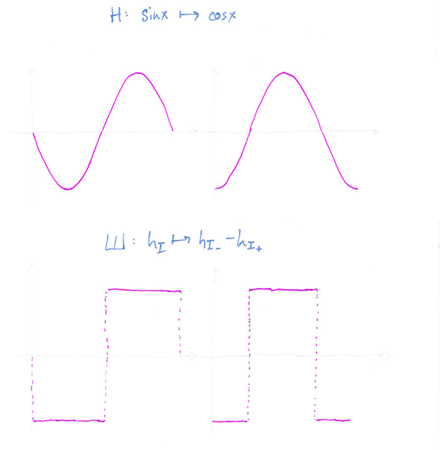
There exists $c \neq 0$ such that

$$c\frac{1}{x-t} = \lim_{L\to\infty} \frac{1}{2\log L} \int_{1/L}^{L} \lim_{R\to\infty} \int_{-R}^{R} K^{\alpha,r}(t,x)\,d\alpha \frac{dr}{r}$$

The Hilbert transform is in the closed convex hull of dyadic shift operators.

The dyadic Hilbert transform

H gives access to harmonic conjugates, $z \mapsto z$ is analytic in $\mathbb{D}$ with boundary values $e^{it} = \cos(t) + i\sin(t)$. So $H : \cos(t) \mapsto \sin(t)$.



UMD spaces

For a Banach space X there holds

$$H : \mathbb{R} \to X \ L^p \text{ bounded if and only if } \ X \text{ is UMD.}$$

UMD means that X-valued martingale difference sequences are unconditional in $L^p(X)$, $1 < p < \infty$.

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The Dyadic shift recovers '$\Rightarrow$' trivially thanks to its proximity to martingale differences.

Part 1

Commutators.

Hankel operators

A Hankel operator with symbol b is an operator of the form

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The commutator operator

$$[H, b] : f \mapsto H(bf) - bH(f)$$

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It is bounded if and only the symbol b is in the space of bounded mean oscillations BMO

$$\|b\|^2_{BMO} = \sup_I \frac{1}{|I|} \int_I \left(b(x) - \frac{1}{|I|} \int_I b \right)^2 dx = \sup_I \langle (b - \langle b \rangle_I)^2 \rangle_I$$

Hankel operators with matrix symbol

When the symbol is a $n \times n$ matrix B and the BMO norm is measured via the so-called strong operator BMO

$$\sup_I \langle \|Be - \langle B \rangle_I e\|^2 \rangle_I \leq \|B\|^2_{BMO} \quad \text{and} \quad \sup_I \langle \|B^* e - \langle B^* \rangle_I e\|^2 \rangle_I \leq \|B\|^2_{BMO}$$

for n-vectors e with $\|e\| = 1$. Then there holds

Theorem (P)

$$\|[H, B] : L^2 \to L^2\| \leq C \log(n) \|B\|_{BMO}$$

This estimate is sharp.

Hankel operators with matrix symbol

The same estimate holds for the matrix Carleson embedding theorem and for the matrix paraproduct. Work by Katz, Nazarov, Pisier, Treil, Volberg.

Indeed, the Haar shift translates commutator estimates into estimates of dyadic paraproducts.

These are operators that arise from the (unbounded) formal product

$$\left(\sum_I (b, h_I) h_I\right) \cdot \left(\sum_J (f, h_J) h_J\right)$$

The commutator with the shift operator induces essential cancellation on the BMO symbol b.

Factorization

If $F \in H^1$, the analytic Hardy space then $F = G_1 G_2$ with $G_i \in H^2$.

Write $F = f + i\tilde{f}$, $G_i = g_i + i\tilde{g_i}$ then

$$Im(G_1 G_2) = \tilde{f} = g_1\tilde{g_2} + \tilde{g_1}g_2$$

Let $b \in BMO$, recall the duality $(H^1)^* = BMO$. Observe that

$$(b, g_1\tilde{g_2} + \tilde{g_1}g_2) = ([b, H]g_1, g_2)$$

Factorization

Let $\|f\|_{H^1} = \|f\|_1 + \|Hf\|_1$ and suppose we have

$$c\|b\|_{BMO} \leq \|[b, H]\|_{2\to 2} \leq C\|b\|_{BMO}$$

If $g_1, g_2 \in L^2$ we can conclude that $f = g_1(Hg_2) + (Hg_1)g_2 \in H^1$

Conversely if $f \in H^1$ then it can be written $f = \sum_i g_i(Hh_i) + (Hg_i)h_i$

Factorization

This is the formulation of (weak) factorization that holds in a variety of settings (real, bi-disk etc) via two-sided commutator estimates.

It uses duality relations in the real variable or the product setting, for example the Chang - Fefferman duality between the multi-parameter real H^1 space of functions $\{f : f, H_1 f, H_2 f \in L^1\}$ and product BMO.

Several parameters

The study of characterisation of multi-parameter BMO spaces and their relationship with commutator bounds was intitiated by Cotlar, Sadosky, Ferguson using the language of Hankel and Toeplitz operators.

It turns out, their theory has deep connections to multi-parameter singular integral theory in real analysis.

Hilbert commutators and BMO

Theorem (Nehari)

$[H, b]$ bounded in L^2 iff $b \in bmo$

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Theorem (Ferguson, Sadosky)

$[H_1 H_2, b]$ *bounded in* L^2 *iff* $b \in bmo$ *'little BMO'*

where $||b||_{bmo} = \max\{\sup_{x_1} ||b(x_1, \cdot)||_{BMO}, \sup_{x_2} ||b(\cdot, x_2)||_{BMO}\}$.

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Theorem (Ferguson, Lacey)

$[H_1, [H_2, b]]$ bounded in L^2 iff $b \in BMO$ 'product BMO'

where $||b||^2_{BMO} = \sup_O \frac{1}{|O|} \sum_{R \subset O} |(b, h_R)|^2$.

Journé commutators

Theorem (Coifman, Rochberg, Weiss)

$[R_i, b]$ bounded in L^2 then $b \in BMO$
$b \in BMO$ then $[T, b]$ bounded in L^2 for Calderón-Zygmund operators.

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Theorem (Lacey, P., Pipher, Wick)

$[R_1^i[R_2^j, b]$ *bounded in* L^2 *then* $b \in BMO$*, product BMO*
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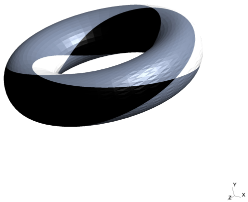
Theorem (Ou, P., Strouse)

$[\otimes R_1^i[\otimes R_2^j, b]$ *bounded in* L^2 *then* $b \in BMO$*, mixed BMO*
$b \in BMO$ *then* $[J_1, [J_2, b]]$ *bounded in* L^2 *for multi-parameter Calderón-Zygmund or Journé operators.*

Journé commutators

Lower estimates: the multi-parameter singular operator that replaces the classical tool of Toeplitz operators has a Fourier multiplier symbol on products of spheres:



Upper estimates: one uses generalisations of the Haar shift, an important result by Hytönen, as well as further generalisations to the product setting by Martikainen and Ou.

Part 2

Weights

Weighted estimates

Define the Hilbert transform using Lebesgue measure dy

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{(x-y)}dy.$$

but measure its L^2 operator norm using a (positive) weight $w \in L^1_{loc}$.

$$\left(\int |Hf(x)|^2 w(x)dx\right)^{1/2} \leq C(w)\left(\int |f(x)|^2 w(x)dx\right)^{1/2}$$

The necessary and sufficient condition for $C(w) < \infty$ is the Muckenhoupt A_2 condition

$$[w]_{A_2} = \sup_I \langle w\rangle_I \langle w^{-1}\rangle_I < \infty$$

where the supremum runs over intervals.

Beurling-Ahlfors

With $1 \leq [w]_{A_2}$ it is of interest to capture the exact growth of the operator with respect to the quantity $[w]_{A_2}$.

In the case of a complex analog of the Hilbert transform and $p > 2$, this solved a regularity problem:

The Beurling-Ahlfors operator is

$$Tf(z) = -p.v.\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(z-\zeta)^2}\, dA(\zeta).$$

The operator T has the property $T \circ \bar{\partial} = \partial$.

Beurling-Ahlfors

Recall that analytic (conformal) functions have zero $\bar{\partial}$ derivatives and map infinitesimal disks to infinitesimal disks.

The Beltrami equation is

$$\bar{\partial} f - \mu \partial f = 0$$

with $\|\mu\|_\infty = k < 1$, it relates to the operator $I - \mu T$ applied to $\bar{\partial} f$.

The significance of $K = (k+1)/(k-1)$ is the ratio of axes of infinitesimal ellipses, images of infinitesimal disks under f, the homeomorphic solution.

Beurling-Ahlfors

What is the minimal requirement of the type $f \in W^{1,p(k)}$ (Sobolev) which guarantees that any solution for a given μ with $\|\mu\|_\infty = k < 1$ self improves to $W^{1,2}$ (hence is continuous)?

Final answer: $p \geq 1 + k$

Astala - Iwaniec - Saksman $p > k + 1$

P. - Volberg $p \geq k + 1$

Beurling-Ahlfors

Using a special weight w related to the equation, then 'any' bound for the operator norm

$$T : L^p(w) \to L^p(w)$$

gives the desired result in the open interval of p since the special weight is in the A_p class:

$$[w]_{A_p} = sup_I \langle w \rangle_I \langle w^{-\frac{1}{p-1}} \rangle_I^{p-1} < \infty$$

Beurling-Ahlfors

The solution of the borderline case required an estimate 'linear' in $[w]_{A_p}$ such as

$$\|T\|_{L^p(w)\to L^p(w)} \leq C_p[w]_{A_p}^{1}, \;\; p > 2$$

because at the borderline, the special weight fails to belong to A_p.

This linear estimate is due to P. - Volberg and was the first of its kind. It follows from the optimal estimate for $p = 2$ via Rubio de Francia's extrapolation.

Nazarov-Treil-Volberg and Wittwer's theorem

The estimate for the Beurling-Ahlfors operator follows through a transference method using the heat equation from the following dyadic model:

$$T_\sigma : h_I \mapsto \sigma_I h_I .$$

Nazarov - Treil - Volberg characterised boundedness in the two-weight setting. Applied to the one-weight case, a theorem by Wittwer states that uniformly in $|\sigma|_\infty \leq 1$,

$$\| T_\sigma \|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}$$

The weighted Hilbert transform

The A_2 conjecture in harmonic analysis was on the Hilbert transform.

When attempting an estimate, one observes obstacles such as integrals of carré du champ type expressions with a shift in time:

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\Delta_I w||\Delta_{I_-} w^{-1}||I|$$

Their (non-split) control using a specific convex function gave the first proof of the A_2 conjecture (2007, P.).

Sparse domination

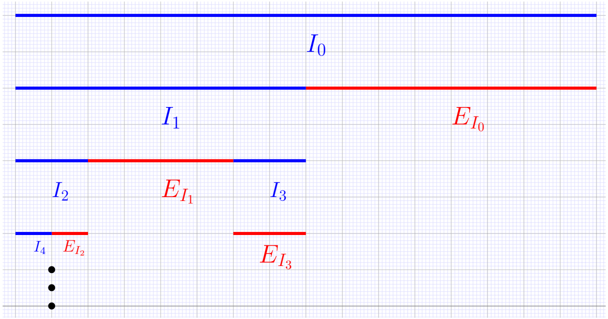
A more recent approach no longer sees the difficulty of the time shift and can get estimates for classical singular operators without the use of shift operators.

For every f there exists a sparse family of intervals $\mathcal{S}$ so that the following domination holds:

$$|Hf(x)| \leq \sum_{I \in \mathcal{S}} \langle |f| \rangle_I \chi_I(x)$$

These results appeared in 2015, Lacey, Lerner - Nazarov.

Sparse collection of intervals



blue intervals form the sparse set and red sets form the disjoint left-overs with a fair share of the mass.

Sparse operator with continuous index

P. - Domelevo develop a sparse domination in a martingale setting in abstract filtered spaces. A sparse operator of a martingale X has an increasing sequence of stopping times $0 = T^{-1} \leq T^{0} \leq \ldots$ with nested sets $O_j = \{T_j < \infty\}$ so that

- $S(X)(\omega) = \sum_{j=-1}^{\infty} |X|_{T^j}(\omega)\chi_{O_j}(\omega)$
- $\forall A^j \subset O_j : A^j \in \mathcal{F}_{T^j} : P(A^j \cap O_{j+1}) \leq \frac{1}{2} P(A^j)$

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If Y and X are martingales under differential subordination, $[X,X]_t - [Y,Y]_t$ non-negative and non-decreasing in t then Y^* is dominated by $S(X)$.

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Aside from weighted estimates for Y^*, one can deduce dimensionless estimates for Riesz vectors in very general settings.

Sparse convex body domination

If f is vector valued and the Hilbert transform applies component wise, there holds the domination via convex bodies. Nazarov - P. - Treil - Volberg showed that

$$Hf(x) \in \sum_{I \in \mathcal{S}} \langle\langle f \rangle\rangle_I \chi_I(x)$$

where

$$\langle\langle f \rangle\rangle_I = \{\langle \phi f \rangle_I : -1 \leq \phi \leq 1\}$$

The matrix A_2 conjecture using the characteristic for positive self-adjoint matrices

$$[W]_{A_2} = sup_I \| \langle W \rangle_I^{1/2} \langle W^{-1} \rangle_I^{1/2} \|^2 < \infty$$

remains open.

Stationary processes

Given a stationary sequence of vectors $e(n), n \in \mathbb{Z}$ in a Hilbert space, that is,

$$(e(n), e(m)) = (e(0), e(m-n)).$$

For example $L^2(\mathbb{T})$ with $e(n) = z^n$. It can be shown that this example is enough by adjusting the measure μ:

$$(e(n), e(m)) = \int z^{n-m} d\mu.$$

Stationary processes

The multivariate version is $e_j(n), n \in \mathbb{Z}, 1 \leq j \leq d$ with

$$(e_j(n), e_k(m)) = (e_j(0), e_k(m-n)).$$

For example $z^m \vec{e}_j, m \in \mathbb{Z}$ and $\{\vec{e}_j : 1 \leq j \leq d\}$ orthonormal base.

This leads to considering matrix measure $M = (\mu_{i,j})$ and $L^2(\mathbb{T}, M)$.

Stationary processes

If we let $E(n) = span\{e_k(n) : 1 \leq k \leq d\}$ then the future is

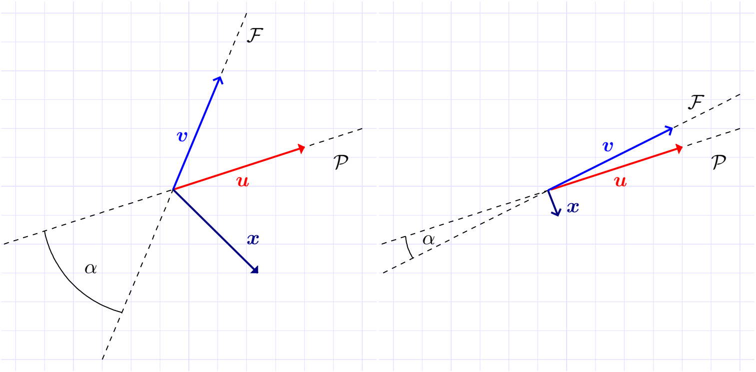
$$\mathcal{F} = span^{cl}\{E(n) : n \geq 0\}$$

and the past

$$\mathcal{P} = span\{E(n) : n < 0\}.$$

The Hilbert transform (or rather its part P_+) anewers questions about the angle between past and future.

Stationary processes



Stationary processes

We have in both cases u and v unit vectors such that $u \in \mathcal{P}$ and $v \in \mathcal{F}$. Let P_+ the projection on $\mathcal{F}$ (the "future"). Take $x = u - v$. We have $P_+ x = -v$, $\|P_+ x\| = 1$, $\|x\|^2 = \|u - v\|^2 = 2 - 2(u, v) = 2(1 - \cos\alpha)$. It follows

$$\|P_+\| \geqslant \frac{\|P_+ x\|}{\|x\|} = \frac{1}{\sqrt{2(1 - \cos\alpha)}}$$

is arbitrarily large when α is small.