Some results on affine Deligne-Lusztig varieties

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Xuhua He

University of Maryland

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(Classical) Deligne-Lusztig varieties

Let G be a connected reductive group over $\mathbb{F}_q$. Let σ be the Frobenius morphism on $G(\overline{\mathbb{F}}_q)$. Let B be a σ-stable Borel subgroup and W be the Weyl group. Then we have the Bruhat decomposition

$$G = \sqcup_{w \in W} BwB.$$

Consider the σ-conjugation action $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$.

Theorem (Lang, '56)

Any two elements in G are σ-conjugate to each other.

Deligne and Lusztig '76 introduced the Deligne-Lusztig variety. For any $w \in W$, set

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Some geometric properties of X_w

- X_w is nonempty;
- $\dim X_w = \ell(w)$;
- [Lusztig, Digne-Michel '06, Bonnafé-Rouquier '06, Görtz '09] The following are equivalent:
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Isocrystals

F a non-arch. local field with valuation ring $\mathcal{O}_F$ and residue field $\mathbb{F}_q$.
$\breve{F}$ completion of F^{un} with valuation ring $\mathcal{O}_{\breve{F}}$ and residue field $k = \overline{\mathbb{F}}_q$.
σ Frobenius morphism of $\breve{F}$ over F.

Definition

An isocrystal is a pair (N, g), where N is a finite dimensional vector space over $\breve{F}$ and $g : N \to N$ is a semi-linear bijection. A crystal M of an isocrystal (N, g) is a g-stable $\mathcal{O}_{\breve{F}}$-lattice of N.

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Isocrystal $\rightsquigarrow$ Newton slope, crystal $\rightsquigarrow$ Hodge slope

Example

Let $N = \oplus_{i=1}^3 \breve{F} e_i$, $g(e_1) = e_2, g(e_2) = e_3, g(e_3) = \epsilon e_1$. Let $M = \oplus_{i=1}^3 \mathcal{O}_{\breve{F}} e_i$ be a crystal of (N, g). Then $\nu(N, g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mu(M) = (1, 0, 0)$.

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Mazur's inequality

Theorem

(1) [Mazur '73] Let (N, g) be an isocrystal and M be a crystal of (N, g). Then

$$\mu(M) \geqslant \nu(N, g).$$

(2) [Kottwitz-Rapoport '03] Let (N, g) be an isocrystal of dimension n. Let $\mu = (a_1, \cdots, a_n) \in \mathbb{Z}^n$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $\mu \geqslant \nu(N, g)$. Then there exists a crystal M of (N, g) with $\mu(M) = \mu$.

Group-theoretic reformulation: $G = GL_n(\breve{F})$ and $K = GL_n(\mathcal{O}_{\breve{F}})$.
Isocrystal $\leftrightsquigarrow$ σ-conjugacy class $[b]$
Crystal $\leftrightsquigarrow$ $K \times K$-double coset $K\mu K$
Mazur's inequality: $K\mu K \cap [b] \neq \emptyset$ if and only if $\nu_b \leqslant \mu$.

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Notation

G reductive group over F, σ Frobenius morphism on $G(\breve{F})$, I σ-stable Iwahori subgroup, $\tilde{W}$ Iwahori-Weyl group. We have

$$G(\breve{F}) = \sqcup_{w \in \tilde{W}} IwI.$$

Note that $\tilde{W}$ is a quasi-Coxeter group, has a natural Bruhat order $\leqslant$ and the length function ℓ on it.
We have $\dim(IwI/I) = \ell(w)$ and $\overline{IwI/I} = \sqcup_{w' \leqslant w} Iw'I/I$.

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Affine Deligne-Lusztig varieties

The affine Deligne-Lusztig varieties we are particularly interested in are

- ADLV in the affine Grassmannian

$$X_\lambda(b) = \{gK \in G/K; g^{-1} b\sigma(g) \in K\epsilon^\lambda K\} \subset Gr.$$

- ADLV in the affine flag variety

$$X_w(b) = \{gI \in G/I; g^{-1} b\sigma(g) \in IwI\} \subset Fl.$$

- The union of ADLV in the affine flag variety

$$X(\mu, b) = \cup_{w \in \mathrm{Adm}(\mu)} X_w(b) \subset Fl.$$

Here the admissible set $\mathrm{Adm}(\mu)$ is defined to be

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W}; w \leqslant t^{\mu'} \text{ for some conjugate } \mu' \text{ of } \mu\}.$$

Major problems

Below we list some major problems on the affine Deligne-Lusztig varieties:

- When is an affine Deligne-Lusztig variety nonempty?
- If it is nonempty, what is its dimension?
- What are the connected components?
- Is there a simple geometric structure for certain affine Deligne-Lusztig varieties?

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Classification of σ-conjugacy classes

Kottwitz '85 & '97 gave a classification of the set $B(G)$ of σ-conjugacy classes of $G(\breve{F})$:

$$(\kappa, \nu) : B(G) \hookrightarrow \pi_1(G)_{\mathrm{Gal}(\bar{F}/F)} \times X_*(T)^+_{\mathbb{Q}},$$

where κ is the Kottwitz map and ν is the Newton map.

A more explicit way: [Görtz-Haines-Kottwitz-Reuman '10, H. '14]
Let $\tilde{W}/_\sigma \tilde{W}$ be the set of σ-conjugacy classes of $\tilde{W}$. Then

$$\tilde{W}/_\sigma \tilde{W} \xrightarrow{\Psi} B(G).$$

Here κ is deduced from the map $\tilde{W} \to \tilde{W}/W_{aff} \cong \pi_1(G)$ and $\nu(w)$ is the unique dominant element conjugate to the rational coweight λ/n, where n is a positive integer with $(w\sigma)^n = t^\lambda$.

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Straight elements

Definition

An element $w \in \tilde{W}$ is called σ-straight if for all $n \in \mathbb{N}$,

$$\ell(w\sigma(w)\cdots\sigma^{n-1}(w)) = n\ell(w).$$

A σ-conjugacy class is called straight if it contains a σ-straight element.

We denote by $\tilde{W}//_\sigma \tilde{W}$ the set of straight σ-conjugacy classes of $\tilde{W}$.

Theorem (H. '14)

$$\tilde{W}//_\sigma \tilde{W} \xleftrightarrow{1-1} B(G).$$

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"Dimension=Degree" Theorem

Theorem (H.-Nie '14)

Let H be the extended affine Hecke algebra of W over $\mathbb{Z}[v^{\pm 1}]$. For any $w \in \tilde{W}$, there exists polynomials $f_{w,\mathcal{O}} \in \mathbb{N}[v - v^{-1}]$ for each σ-conjugacy class $\mathcal{O}$ such that

$$T_w \equiv \sum f_{w,\mathcal{O}} T_{w_{\mathcal{O}}} \mod [H, H]_\sigma,$$

here $w_{\mathcal{O}}$ is a minimal length representative of $\mathcal{O}$.

Theorem (H. '14)

Let $b \in G(L)$ and $w \in \tilde{W}$. Then

$$\dim X_w(b) = \max_{\mathcal{O}; \kappa(\mathcal{O}) = \kappa(b), \nu(\mathcal{O}) = \nu(b)} \frac{1}{2}\Big(\ell(w) + \ell(w_{\mathcal{O}}) + \deg f_{w,\mathcal{O}}\Big) - \langle \bar{\nu}_b, 2\rho \rangle.$$

Moreover, $X_w(b) \neq \emptyset$ iff $f_{w,\mathcal{O}} \neq 0$ for some $\mathcal{O}$ with $(\kappa, \nu)(\mathcal{O}) = (\kappa, \nu)(b)$.

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Nonemptiness pattern

Theorem (Rapoport-Richartz '96, Kottwitz '03, Gashi '10)

Let λ be a dominant coweight and $b \in G$. Then $X_\lambda(b) \neq \emptyset$ if and only if $\kappa([b]) = \kappa(\lambda)$ and $\nu_b \leq \lambda$.

Theorem (Görtz-Haines-Kottwitz-Reuman '10, Görtz-H.-Nie '15)

Let G be a quasi-split group and $[b] \in B(G)$ be basic. Then $X_w(b) \neq \emptyset$ if and only if there is no "Levi obstruction".

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Let $[b] \in B(G)$. Then $X(\mu, b) \neq \emptyset$ if and only if $[b] \in B(G, \mu)$, i.e. $\kappa([b]) = \kappa(\mu)$ and ν_b is less than or equal to the Galois average of μ.

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Dimension formula

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Let λ be a dominant coweight and $b \in G$. If $X_\lambda(b) \neq \varnothing$, then

$$\dim X_\lambda(b) = \langle \lambda - \nu_b, \rho \rangle - \frac{1}{2} def_G(b),$$

where $def_G(b)$ is the defect of b.

Theorem (Conjecture of Görtz-Haines-Kottwitz-Reuman '10, H. '14 & '16)

Let $[b] \in B(G)$ be basic and $w \in W$ be an element in the shrunken Weyl chamber (i.e., the lowest two-sided cell of W). If $X_w(b) \neq \varnothing$, then

$$\dim X_w(b) = \frac{1}{2}(\ell(w) + \ell(\eta_\sigma(w)) - def_G(b)).$$

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Example of G_2

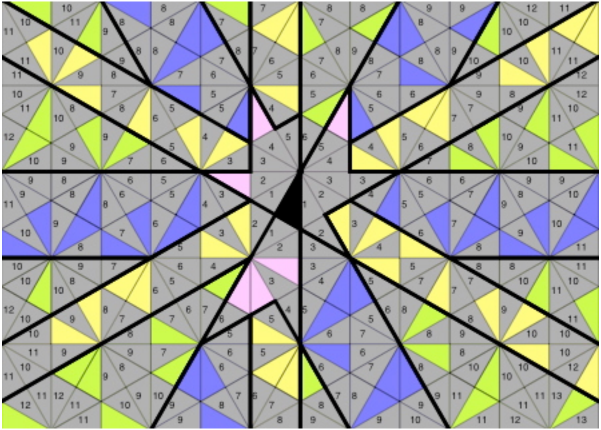

Figure: Görtz-Haines-Kottwitz-Reuman, "Dimensions of some affine Deligne-Lusztig varieties", Ann. Sci. École Norm. Sup. (2006).

Hodge-Newton decomposition

The pair (λ, b) is called Hodge-Newton decomposable w.r.t. a proper Levi subgroup M if $b \in M$ and $\kappa_M(\lambda) = \kappa_M(b)$.

Theorem (Katz '79, Kottwitz '03)

Let (λ, b) be Hodge-Newton decomposable w.r.t. a Levi M. Then

$$X_\lambda^M(b) \cong X_\lambda^G(b).$$

Theorem (Görtz-H.-Nie)

Suppose that (μ, b) is Hodge-Newton decomposable with respect to some proper Levi subgroup. Then

$$X(\mu, b) \cong \bigsqcup_{P'=M'N'} X^{M'}(\mu_{P'}, b_{P'}),$$

where P' runs through a certain finite set of semistandard parabolic subgroups. The subsets in the union are open and closed.

Connected components

Theorem (Viehmann '08, Chen-Kisin-Viehmann '15, Nie)

Assume that G is an unramified simple group and that (λ, b) is Hodge-Newton indecomposable. Then

$$\pi_0(X_{\preceq\lambda}(b)) \cong \pi_1(G)^{\sigma}_{\Gamma_0}.$$

Theorem (H.-Zhou)

Assume that $[b] \in B(G, \mu)$ is basic and that (μ, b) is Hodge-Newton indecomposable. Then $\pi_0(X(\mu, b)) \cong \pi_1(G)^{\sigma}_{\Gamma_0}$.

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Theorem (Chen-Nie)

Let G be a split group. Assume that $[b] \in B(G,\mu)$ and that (μ, b) is Hodge-Newton indecomposable. Then $\pi_0(X(\mu, b)) \cong \pi_1(G)$.

Connected components

Theorem (Viehmann '08, Chen-Kisin-Viehmann '15, Nie)

Assume that G is an unramified simple group and that (λ, b) is Hodge-Newton indecomposable. Then

$$\pi_0(X_{\preceq\lambda}(b)) \cong \pi_1(G)^{\sigma}_{\Gamma_0}.$$

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ADLV with simple geometric structure

Theorem (Görtz-H.-Nie)

Assume that G is simple, μ is a dominant coweight of G and K' is a parahoric subgroup. Then the following conditions are equivalent:

- *For basic $[b_0] \in B(G, \mu)$, $X(\mu, b_0)_{K'}$ is naturally a union of classical Deligne-Lusztig varieties;*
- *For any nonbasic $[b] \in B(G, \mu)$, $\dim X(\mu, b)_{K'} = 0$;*
- *The pair (μ, b) is Hodge-Newton decomposable for any nonbasic $[b] \in B(G, \mu)$;*
- *The coweight μ is minute for G.*

For quasi-split groups, minute means that $\sum_{i \in \mathcal{O}} \langle \mu, \omega_i^\vee \rangle \leqslant 1$ for any Galois orbit $\mathcal{O}$ on the set of simple roots. The definition for non quasi-split group is more involved.

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Classification

Fully Hodge-Newton decomposable cases are

- The fake unitary case. Studied by Harris-Taylor.
- The Drinfeld case.
- $U(1, n)$ of a hermitian form. Studied by Vollaard-Wedhorn '11 (unramified case) and by Rapoport-Terstiege-Wilson '14 (ramified case).
- The Hilbert-Blumenthal case. Studied by Tian-Xiao '16.
- $SO(2, n)$ of a quadratic form. Studied by Howard-Pappas '17.
- A few exceptional cases.

Application to Shimura varieties

Theorem (H.-Rapoport '17)

Under several axioms of Shimura varieties,

- *The Kottwitz-Rapoport stratum* $KR_{K,w}$ *is nonempty if and only if* $w \in Adm(\mu)_K$.
- *The Newton stratum* $N_{K,[b]}$ *is nonempty if and only if* $[b] \in B(G, \mu)$.

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$$Sh_K = \sqcup_{w \in Adm(\mu) \cap {}^K \tilde{W}} EKOR_{K,w}.$$

The closure of a EKOR stratum is a union of EKOR strata.

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Some other applications

In below, we briefly mention some other applications of ADLV.

- Rad-Hartl established the Langlands-Rapoport conjecture over function field, in which the nonemptiness of $X(\mu, b)$ is used.
- The work of Kisin '17, and Zhou on the Langlands-Rapoport conjecture for Shimura varieties, in which the description of the connected components of $X(\mu, b)$ is used.
- Chen-Fargues-Shen established the Fargues-Rapoport conjecture weakly admissible=admissible iff it is fully HN decomposable.
- The work of Rapoport-Terstiege-Zhang '13 and Li-Zhu '17 towards Zhang's AFL, in which the basic locus of unramified $U(1, n)$ is used.
- The work of Helm-Tian-Xiao '17 on the Tate conjecture for certain Shimura varieties, in which the basic locus of ADLV in the Hilbert-Blumenthal case is used.

Open problems

We mention some open problems on the affine Deligne-Lusztig varieties.

- The nonemptiness pattern of $X_w(b)$ for nonbasic b, in particular, the asymptotic behavior for w. [conjectured by Görtz-Haines-Kottwitz-Reuman]
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