Let $G$ be a connected reductive group over $\mathbb{F}_q$. Let $\sigma$ be the Frobenius morphism on $G(\overline{\mathbb{F}_q})$. Let $B$ be a $\sigma$-stable Borel subgroup and $W$ be the Weyl group. Then we have the Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB.$$ 

Consider the $\sigma$-conjugation action $g \cdot_{\sigma} g' = gg'\sigma(g)^{-1}$.

**Theorem (Lang, ’56)**

*Any two elements in $G$ are $\sigma$-conjugate to each other.*

Deligne and Lusztig ’76 introduced the Deligne-Lusztig variety. For any $w \in W$, set

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(Classical) Deligne-Lusztig varieties

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- $X_w$ is nonempty;
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Isocrystals

$F$ a non-arch. local field with valuation ring $\mathcal{O}_F$ and residue field $\overline{\mathbb{F}}_q$.
$	ilde{F}$ completion of $F^{un}$ with valuation ring $\mathcal{O}_{\tilde{F}}$ and residue field $k = \overline{\mathbb{F}}_q$.
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An isocrystal is a pair $(N, g)$, where $N$ is a finite dimensional vector space over $\tilde{F}$ and $g : N \to N$ is a semi-linear bijection. A crystal $M$ of an isocrystal $(N, g)$ is a $g$-stable $\mathcal{O}_{\tilde{F}}$-lattice of $N$. 

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Isocrystal \( \leadsto \) Newton slope, crystal \( \leadsto \) Hodge slope

Example

Let \( N = \bigoplus_{i=1}^3 \bar{F}e_i \), \( g(e_1) = e_2 \), \( g(e_2) = e_3 \), \( g(e_3) = \epsilon e_1 \). Let \( M = \bigoplus_{i=1}^3 \mathcal{O}_{\bar{F}}e_i \) be a crystal of \( (N, g) \). Then \( \nu(N, g) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) and \( \mu(M) = (1, 0, 0) \).
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Mazur’s inequality

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(1) [Mazur ’73] Let \((N, g)\) be an isocrystal and \(M\) be a crystal of \((N, g)\). Then

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\mu(M) \geq \nu(N, g).
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(2) [Kottwitz-Rapoport ’03] Let \((N, g)\) be an isocrystal of dimension \(n\). Let \(\mu = (a_1, \ldots, a_n) \in \mathbb{Z}^n\) with \(a_1 \geq a_2 \geq \cdots \geq a_n\) and \(\mu \geq \nu(N, g)\). Then there exists a crystal \(M\) of \((N, g)\) with \(\mu(M) = \mu\).

Group-theoretic reformulation: \(G = \text{GL}_n(\bar{\mathcal{F}})\) and \(K = \text{GL}_n(\mathcal{O}_F)\).

Isocrystal \(\leftrightarrow\) \(\sigma\)-conjugacy class \([b]\)

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Notation

$G$ reductive group over $F$, $\sigma$ Frobenius morphism on $G(\overline{F})$, $I$ $\sigma$-stable Iwahori subgroup, $\tilde{W}$ Iwahori-Weyl group. We have

$$G(\overline{F}) = \bigsqcup_{w \in \tilde{W}} Iwl.$$

Note that $\tilde{W}$ is a quasi-Coxeter group, has a natural Bruhat order $\leq$ and the length function $\ell$ on it.

We have $\dim(Iwl/I) = \ell(w)$ and $Iwl/I = \bigsqcup_{w' \leq w} Iw'I/I$. 

If $G$ is unramified, we let $K \supset I$ be a hyperspecial parahoric subgroup. Then

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The affine Grassmannian $Gr = G(\tilde{F})/K$
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The affine flag variety $Fl = G(\tilde{F})/I$

The affine Grassmannian $Gr = G(\tilde{F})/K$
The affine Deligne-Lusztig varieties we are particularly interested in are

- **ADLV in the affine Grassmannian**

\[ X_{\lambda}(b) = \{ gK \in G/K ; g^{-1}b\sigma(g) \in K\epsilon^\lambda K \} \subset Gr. \]

- **ADLV in the affine flag variety**

\[ X_w(b) = \{ gl \in G/l ; g^{-1}b\sigma(g) \in lwI \} \subset Fl. \]

- The union of ADLV in the affine flag variety

\[ X(\mu, b) = \bigcup_{w \in \text{Adm}(\mu)} X_w(b) \subset Fl. \]

Here the admissible set \( \text{Adm}(\mu) \) is defined to be

\[ \text{Adm}(\mu) = \{ w \in \tilde{W} ; w \leq t^{\mu'} \text{ for some conjugate } \mu' \text{ of } \mu \}. \]
Below we list some major problems on the affine Deligne-Lusztig varieties:

- When is an affine Deligne-Lusztig variety nonempty?
- If it is nonempty, what is its dimension?
- What are the connected components?
- Is there a simple geometric structure for certain affine Deligne-Lusztig varieties?

These are interesting, yet challenging problems in Lie theory and have important applications to number theory.
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A major difference between affine Deligne-Lusztig varieties and classical Deligne-Lusztig varieties is that affine Deligne-Lusztig varieties have the second parameter: the $\sigma$-conjugacy class $[b]$ of $G(\bar{F})$. 
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Classification of $\sigma$-conjugacy classes

Kottwitz ’85 & ’97 gave a classification of the set $B(G)$ of $\sigma$-conjugacy classes of $G(\tilde{F})$:

$$(\kappa, \nu): B(G) \hookrightarrow \pi_1(G)_{\text{Gal} (\tilde{F}/F)} \times X_*(T)^+_Q,$$

where $\kappa$ is the Kottwitz map and $\nu$ is the Newton map.

A more explicit way: [Görtz-Haines-Kottwitz-Reuman ’10, H. ’14]

Let $\tilde{W}/_\sigma \tilde{W}$ be the set of $\sigma$-conjugacy classes of $\tilde{W}$. Then

$$\tilde{W}/_\sigma \tilde{W} \xrightarrow{\Psi} B(G).$$

Here $\kappa$ is deduced from the map $\tilde{W} \to \tilde{W}/W_{aff} \cong \pi_1(G)$ and $\nu(w)$ is the unique dominant element conjugate to the rational coweight $\lambda/n$, where $n$ is a positive integer with $(w\sigma)^n = t^\lambda$. 
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Straight elements

Definition
An element $w \in \tilde{W}$ is called $\sigma$-straight if for all $n \in \mathbb{N}$,

$$\ell(w \sigma(w) \cdots \sigma^{n-1}(w)) = n \ell(w).$$

A $\sigma$-conjugacy class is called straight if it contains a $\sigma$-straight element.

We denote by $\tilde{W} // \sigma \tilde{W}$ the set of straight $\sigma$-conjugacy classes of $\tilde{W}$.

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“Dimension=Degree” Theorem

Theorem (H.-Nie ’14)

Let $H$ be the extended affine Hecke algebra of $W$ over $\mathbb{Z}[\nu^{\pm 1}]$. For any $w \in \tilde{W}$, there exists polynomials $f_{w,\mathcal{O}} \in \mathbb{N}[\nu - \nu^{-1}]$ for each $\sigma$-conjugacy class $\mathcal{O}$ such that

$$T_w \equiv \sum f_{w,\mathcal{O}} T_{w_{\mathcal{O}}} \mod [H, H]_{\sigma},$$

here $w_{\mathcal{O}}$ is a minimal length representative of $\mathcal{O}$.

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Let $b \in G(L)$ and $w \in \tilde{W}$. Then

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Moreover, $X_w(b) \neq \emptyset$ iff $f_{w,\mathcal{O}} \neq 0$ for some $\mathcal{O}$ with $(\kappa, \nu)(\mathcal{O}) = (\kappa, \nu)(b)$. 
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Theorem (Rapoport-Richartz ’96, Kottwitz ’03, Gashi ’10)

Let $\lambda$ be a dominant coweight and $b \in G$. Then $X_\lambda(b) \neq \emptyset$ if and only if $\kappa([b]) = \kappa(\lambda)$ and $\nu_b \leq \lambda$.

Theorem (Görtz-Haines-Kottwitz-Reuman ’10, Görtz-H.-Nie ’15)

Let $G$ be a quasi-split group and $[b] \in B(G)$ be basic. Then $X_w(b) \neq \emptyset$ if and only if there is no “Levi obstruction”.

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Let $\lambda$ be a dominant coweight and $b \in G$. If $X_\lambda(b) \neq \emptyset$, then

$$\dim X_\lambda(b) = \langle \lambda - \nu_b, \rho \rangle - \frac{1}{2} \text{def}_G(b),$$

where $\text{def}_G(b)$ is the defect of $b$.

Theorem (Conjecture of Görtz-Haines-Kottwitz-Reuman ’10, H. ’14 & ’16)

Let $[b] \in B(G)$ be basic and $w \in W$ be an element in the shrunken Weyl chamber (i.e., the lowest two-sided cell of $W$). If $X_w(b) \neq \emptyset$, then

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Example of $G_2$

Hodge-Newton decomposition

The pair \((\lambda, b)\) is called Hodge-Newton decomposable w.r.t. a proper Levi subgroup \(M\) if \(b \in M\) and \(\kappa_M(\lambda) = \kappa_M(b)\).

**Theorem (Katz '79, Kottwitz '03)**

Let \((\lambda, b)\) be Hodge-Newton decomposable w.r.t. a Levi \(M\). Then

\[
X^M_\chi(b) \cong X^G_\chi(b).
\]

**Theorem (Görtz-H.-Nie)**

Suppose that \((\mu, b)\) is Hodge-Newton decomposable with respect to some proper Levi subgroup. Then

\[
X(\mu, b) \cong \bigsqcup_{P' = M' N'} X^{M'}(\mu_{P'}, b_{P'}),
\]

where \(P'\) runs through a certain finite set of semistandard parabolic subgroups. The subsets in the union are open and closed.
Connected components

**Theorem (Viehmann ’08, Chen-Kisin-Viehmann ’15, Nie)**

Assume that $G$ is an unramified simple group and that $(\lambda, b)$ is Hodge-Newton indecomposable. Then

$$\pi_0(X_{\leq \lambda}(b)) \cong \pi_1(G)^\sigma_{\Gamma_0}.$$ 

**Theorem (H.-Zhou)**

Assume that $[b] \in B(G, \mu)$ is basic and that $(\mu, b)$ is Hodge-Newton indecomposable. Then

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Let $G$ be a split group. Assume that $[b] \in B(G, \mu)$ and that $(\mu, b)$ is Hodge-Newton indecomposable. Then $\pi_0(X(\mu, b)) \cong \pi_1(G).$
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Theorem (Görtz-H.-Nie)

Assume that $G$ is simple, $\mu$ is a dominant coweight of $G$ and $K'$ is a parahoric subgroup. Then the following conditions are equivalent:

- For basic $[b_0] \in B(G, \mu)$, $X(\mu, b_0)_{K'}$ is naturally a union of classical Deligne-Lusztig varieties;
- For any nonbasic $[b] \in B(G, \mu)$, $\dim X(\mu, b)_{K'} = 0$;
- The pair $(\mu, b)$ is Hodge-Newton decomposable for any nonbasic $[b] \in B(G, \mu)$;
- The coweight $\mu$ is minute for $G$.

For quasi-split groups, minute means that $\sum_{i \in \mathcal{O}} \langle \mu, \omega_i^\vee \rangle \leq 1$ for any Galois orbit $\mathcal{O}$ on the set of simple roots. The definition for non quasi-split group is more involved.
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Classification

Fully Hodge-Newton decomposable cases are

- The fake unitary case. Studied by Harris-Taylor.
- The Drinfeld case.
- $U(1, n)$ of a hermitian form. Studied by Vollaard-Wedhorn ‘11 (unramified case) and by Rapoport-Terstiege-Wilson ‘14 (ramified case).
- $SO(2, n)$ of a quadratic form. Studied by Howard-Pappas ‘17.
- A few exceptional cases.
Theorem (H.-Rapoport ’17)

Under several axioms of Shimura varieties,

- The Kottwitz-Rapoport stratum $KR_{K,w}$ is nonempty if and only if $w \in \text{Adm}(\mu)_K$.
- The Newton stratum $N_{K,[b]}$ is nonempty if and only if $[b] \in B(G,\mu)$. 

There exists the Ekedahl-Kottwitz-Oort-Rapoport stratification on $Sh_K$ with arbitrary parahoric level structure, $Sh_K = \bigcup_{w \in \text{Adm}(\mu)} \bigcup \tilde{W}_{EKOR}$, $w$. 

The closure of a EKOR stratum is a union of EKOR strata.
Application to Shimura varieties

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- (Grothendieck’s conjecture) For $[b], [b'] \in B(G,\mu)$,

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Some other applications

In below, we briefly mention some other applications of ADLV.

- Rad-Hartl established the Langlands-Rapoport conjecture over function field, in which the nonemptiness of $X(\mu, b)$ is used.
- The work of Kisin ’17, and Zhou on the Langlands-Rapoport conjecture for Shimura varieties, in which the description of the connected components of $X(\mu, b)$ is used.
- Chen-Fargues-Shen established the Fargues-Rapoport conjecture weakly admissible=admissible iff it is fully HN decomposable.
- The work of Rapoport-Terstiege-Zhang ’13 and Li-Zhu ’17 towards Zhang’s AFL, in which the basic locus of unramified $U(1, n)$ is used.
- The work of Helm-Tian-Xiao ’17 on the Tate conjecture for certain Shimura varieties, in which the basic locus of ADLV in the Hilbert-Blumenthal case is used.
Open problems

We mention some open problems on the affine Deligne-Lusztig varieties.

- The nonemptiness pattern of $X_w(b)$ for nonbasic $b$, in particular, the asymptotic behavior for $w$. [conjectured by Görtz-Haines-Kottwitz-Reuman]

- The dimension of $X_w(b)$ for basic $b$ and $w$ in the critical strip. Little is known.
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IGNORAMUS

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