Double affine Grassmannians and Coulomb branches of $3d\ \mathcal{N} = 4$ quiver gauge theories

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Hall-Littlewood functions

In order to compute spherical functions for p-adic groups, Macdonald introduced Hall-Littlewood functions for a semisimple Lie group H:

$$P_\lambda(q) = \frac{1}{W_\lambda(q)} \frac{\sum_{w \in W} (-1)^{\ell(w)} w \left(e^{\lambda+\rho} \prod_{\alpha \in R^+} (1 - qe^{-\alpha}) \right)}{e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})},$$

where $\lambda \in \Lambda^+$ is a dominant weight of H,
(R the root system; W the Weyl group, and $W_\lambda(q)$ the Poincaré polynomial of the stabilizer of λ in W).

- $P_\lambda(0) = \chi^\lambda$, the irreducible characters.

Kostka-Foulkes polynomials

- The Kostka-Foulkes polynomials $K_{\lambda\mu}(q)$:
the matrix entries of the transformation matrix to the basis of irreducible characters χ^λ from the Hall-Littlewood basis of symmetric functions:

$$\chi^\lambda = \sum_{\Lambda^+ \ni \mu \leq \lambda} K_{\lambda\mu}(q) P_\mu(q).$$

- Lusztig: $K_{\lambda\mu}$ are particular cases of Kazhdan-Lusztig polynomials for the affine Weyl group $\Rightarrow K_{\lambda\mu}(q) \in \mathbb{N}[q]$.
Also, $K_{\lambda\mu}(1) = \dim V_\mu^\lambda$ is the weight multiplicity.
So $K_{\lambda\mu}(q)$ is a q-analogue of the weight multiplicity.

Geometric meaning

The positivity property $K_{\lambda\mu}(q) \in \mathbb{N}[q]$ holds due to the geometric interpretation as a Poincaré polynomial:

$$q^{-\dim \mathrm{Gr}^\mu_G/2} K_{\lambda\mu}(q^{-1}) = \sum_{i>0} \dim \mathcal{IC}(\overline{\mathrm{Gr}}^\lambda_G)^{-2i}_\mu q^{-i},$$

where G is the Langlands dual of our initial group H, i.e. $H = G^\vee$,

- $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$ is the affine Grassmannian of G $(\mathcal{O} = \mathbb{C}[\![z]\!],\ \mathcal{K} = \mathbb{C}(\!(z)\!))$. It is the union of $G_\mathcal{O}$-orbits Gr^λ_G numbered by the dominant coweights of G, i.e. dominant weights $\lambda \in \Lambda^+$ of $H = G^\vee$. The closure $\overline{\mathrm{Gr}}^\lambda_G \subset \mathrm{Gr}_G$ of a $G_\mathcal{O}$-orbit Gr^λ_G is the union $\bigsqcup_{\Lambda^+ \ni \mu \leq \lambda} \mathrm{Gr}^\mu_G$. Finally, $\mathcal{IC}(\overline{\mathrm{Gr}}^\lambda_G)^{-2i}_\mu$ is the cohomology in degree $-2i$ of the stalk of the IC-sheaf at a point in Gr^μ_G.

Geometric Satake equivalence

This geometric interpretation of the q-analogue of weight multiplicities is one of manifestations of the geometric Satake equivalence: a tensor equivalence of $\mathrm{Rep}(G^\vee), \otimes$ and the abelian category $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ of $G_\mathcal{O}$-equivariant perverse sheaves on Gr_G, equipped with the convolution monoidal structure $\star$.

This tensor equivalence (Lusztig, Drinfeld, Ginzburg, Beilinson, Mirković, Vilonen,...) is a cornerstone of the geometric Langlands program.

Satake isomorphism

It is a categorification of the following classical result. If the base field is $\mathbb{F}_q$, and the coefficient field is $\mathbb{Q}_\ell$, then the convolution algebra of $G_\mathcal{O}$-invariant functions on Gr_G (spherical affine Hecke algebra) is isomorphic to $\mathbb{Q}_\ell[G^\vee]^{G^\vee}$ (Satake isomorphism).

The characteristic function of a $G_\mathcal{O}$-orbit Gr_G^λ is P_λ, while the Frobenius trace function of $\mathcal{IC}(\overline{\mathrm{Gr}}_G^\lambda)$ is χ^λ.

Kac-Moody Lie algebras

- For a symmetrizable Kac-Moody Lie algebra $\mathfrak{h}$ with the root system R and the Weyl group W, Viswanath defined the Hall-Littlewood functions

$$P_\lambda(q) = \frac{1}{W_\lambda(q)} \frac{\sum_{w\in W}(-1)^{\ell(w)} w\left(e^{\lambda+\rho}\prod_{\alpha\in R^+}(1-qe^{-\alpha})^{m_\alpha}\right)}{e^\rho \prod_{\alpha\in R^+}(1-e^{-\alpha})^{m_\alpha}},$$

where $\lambda \in \Lambda^+$ is a dominant weight, and m_α is the multiplicity of a positive root α.

- Again, the Kostka-Foulkes polynomials are defined via $\sum_{\Lambda^+\ni\mu\leq\lambda} K_{\lambda\mu}(q)P_\mu(q) = \chi^\lambda$ (irreducible characters).

Affine Lie algebras

Viswanath proved $K_{\lambda\mu}(q) \in \mathbb{Z}[q]$ and suggested

Conjecture $K_{\lambda\mu}(q) \in \mathbb{N}[q]$.

Question What is a geometric meaning of $K_{\lambda\mu}(q)$?

Example $\mathfrak{h}$ a dual untwisted affine Lie algebra, with the Langlands dual (i.e. with the transposed Cartan matrix) affine Lie algebra $\mathfrak{g}_{\mathsf{aff}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$.
λ the basic fundamental weight of $\mathfrak{h}$ (at level 1),
$\mu = \lambda - a\delta$ for the minimal imaginary root δ; $a \in \mathbb{N}$.
G the simply connected almost simple Lie group with Lie algebra $\mathfrak{g}$.

Uhlenbeck space

$\mathrm{Bun}_G^a(\mathbb{A}^2)$ is the moduli space of G-bundles on $\mathbb{P}^2$, with the second Chern class a, equipped with a trivialization at the infinite line $\mathbb{P}^1_\infty \subset \mathbb{P}^2$.
In other words, $\mathrm{Bun}_G^a(\mathbb{A}^2)$ is the moduli space of G_c-instantons on $\mathbb{R}^4$ with topological charge a. It possesses a (partial) Uhlenbeck compactification $\mathcal{U}_G^a(\mathbb{A}^2) = \bigsqcup_{0\leq b\leq a} \mathrm{Bun}_G^b(\mathbb{A}^2) \times \mathrm{Sym}^{a-b}(\mathbb{A}^2)$ (the moduli space of ideal instantons). We have

$$K_{\lambda\mu}(q^{-1}) = \sum_{i>0} \dim \mathcal{IC}(\mathcal{U}_G^a(\mathbb{A}^2))_0^{-2i} q^{-i},$$

where $0 = a \cdot 0 \in \mathrm{Sym}^a(\mathbb{A}^2) \subset \mathcal{U}_G^a(\mathbb{A}^2)$.

Double affine Grassmannian

Based on this equality, I. Frenkel (and also I. Grojnowski) suggested that the Uhlenbeck spaces should play a role of affine Grassmannian for affine Lie algebras (i.e. double affine Grassmannian).

Note a discrepancy: $\overline{\mathrm{Gr}}_G^\lambda$ is a projective variety, while $\mathcal{U}_G^a(\mathbb{A}^2)$ is affine. The reason is $\overline{\mathrm{Gr}}_G^\lambda = \bigsqcup_{\Lambda^+ \ni \mu \leq \lambda} \mathrm{Gr}_G^\mu$ is a finite union since Λ^+ has the minimal elements (0 and minuscule coweights).

But for a Kac-Moody algebra of nonfinite type Λ^+ does not have minimal elements. So while Gr_G is a union of projective varieties $\overline{\mathrm{Gr}}_G^\lambda$, the sought-for affine Grassmannian for a general Kac-Moody algebra must be of semiinfinite nature.

Transversal slices

So the best we can hope for in general case are the transversal slices $\overline{\mathcal{W}}^\lambda_\mu,\ \mu \leq \lambda$.

In the finite case $\overline{\mathcal{W}}^\lambda_\mu$ is the intersection of $\overline{\mathrm{Gr}}^\lambda_G$ and the opposite orbit $K_1 \cdot \mu$ of the congruence subgroup $K_1 \subset G[z^{-1}]$, the kernel of evaluation $G[z^{-1}] \to G$.

Due to transversality, $\mathcal{IC}(\overline{\mathcal{W}}^\lambda_\mu)^\bullet_\mu = \mathcal{IC}(\overline{\mathrm{Gr}}^\lambda_G)^\bullet_\mu$ (up to a shift) is a graded version of the weight space V^λ_μ.

- $\mathcal{W}^\lambda_\mu := \mathrm{Gr}^\lambda_G \cap (K_1 \cdot \mu)$ are the symplectic leaves of a natural Poisson structure on Gr_G (Mirković, Kamnitzer, Webster, Weekes, Yacobi).

Nakajima quiver varieties

Another geometric realization of the weight spaces of integrable representations for symmetric Kac-Moody Lie algebras: via Nakajima quiver varieties.

The (off-diagonal part of the) Cartan matrix is viewed as the incidence matrix of a graph. We choose an orientation and view it as a quiver Q with the set of vertices Q_0 and the set of arrows Q_1.

Given Q_0-graded vector spaces $V = \oplus_{i\in Q_0} V_i$ and $W = \oplus_{i\in Q_0} W_i$, we set $GL(V) := \prod_{i\in Q_0} GL(V_i)$, and consider its representation $\mathbf{N} := \bigoplus_{e\in Q_1} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}(W_i, V_i)$.

Nakajima quiver varieties

The cotangent space $\mathbf{N} \oplus \mathbf{N}^*$ carries a natural symplectic form and a symplectic action of $GL(V)$. Consider the Hamiltonian reduction

$$\mathfrak{M}_0(V, W) = (\mathbf{N} \oplus \mathbf{N}^*) /\!\!/\!\!/ GL(V)$$

(the spectrum of the ring of $GL(V)$-invariant functions on the zero level of the moment map), and also its GIT version

$$\mathfrak{M}(V, W) = (\mathbf{N} \oplus \mathbf{N}^*) /\!\!/\!\!/_{\det} GL(V)$$

(the projective spectrum of the ring of semiinvariants with respect to the character $\prod_{i \in Q_0} \det \colon GL(V) \to \mathbb{C}^\times$).

- The natural projection $\pi \colon \mathfrak{M}(V, W) \to \mathfrak{M}_0(V, W)$ is a semismall resolution of singularities under the following numerical conditions.

Recall that Q_0 is the set of simple roots / fundamental weights of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$. We set

$$\lambda = \sum_{i \in Q_0} \dim(W_i)\omega_i,\ \alpha = \sum_{i \in Q_0} \dim(V_i)\alpha_i,\ \mu = \lambda - \alpha.$$

Theorem (Nakajima) (a) If μ is dominant and enters with nonzero mutiplicity into the irreducible integrable representation V^λ of $\mathfrak{g}$, then $\pi\colon \mathfrak{M}(V,W) \to \mathfrak{M}_0(V,W)$ is a semismall resolution of singularities.
(b) Under the conditions of (a), there is a natural isomorphism $H_{\mathrm{top}}(\pi^{-1}(0)) \simeq V^\lambda_\mu$. In other words, a basis in V^λ_μ is formed by the classes of irreducible components of the Lagrangian subvariety $\pi^{-1}(0) \subset \mathfrak{M}(V,W)$.

Thus in the finite case, the weight components V^λ_μ of irreducible representations arise from the geometry of both $\overline{\mathcal{W}}^\lambda_\mu$ and $\mathfrak{M}_0(V, W)$, but in two different ways.

Such a relation between (singular) symplectic varieties $\mathfrak{M}_0(V, W)$ and $\overline{\mathcal{W}}^\lambda_\mu$ is an instance of the so called symplectic duality (Braden, Licata, Proudfoot, Webster):

X an affine normal conical Poisson variety. We assume X to be singular symplectic, with a symplectic resolution $\pi\colon \widetilde{X} \to X$. We set

- $\mathfrak{s}_X := H^2(\widetilde{X}, \mathbb{C})$.
- $\mathfrak{t}_X$ is the Lie algebra of a Cartan torus in the group of symplectomorphisms of X commuting with the contracting $\mathbb{C}^\times$-action.

Symplectic duality

Sometimes one can find another such variety $X^\vee$ with isomorphisms $\mathfrak{s}_X \simeq \mathfrak{t}_{X^\vee}$, $\mathfrak{t}_X \simeq \mathfrak{s}_{X^\vee}$.

Examples. 1. X and $X^\vee$ the nilpotent cones in the Langlands dual Lie algebras.

2. For partitions $\lambda \geq \mu$ of n, let $\mathcal{S}^\lambda_\mu$ be the intersection of the nilpotent orbit closure $\overline{\mathbb{O}}_\lambda \subset \mathfrak{gl}(n)$ with the Slodowy slice to the orbit $\mathbb{O}_\mu$. Then $\mathcal{S}^\lambda_\mu$ is dual to $\mathcal{S}^{\mu^t}_{\lambda^t}$.

3. (Gale) duality of toric hyperkähler manifolds.

4. $\mathrm{Sym}^a(\mathbb{A}^2/\Gamma)^\vee \simeq \mathcal{U}^a_G(\mathbb{A}^2)/\mathbb{G}^2_a$ for a finite subgroup $\Gamma \subset SL(2)$ corresponding by McKay to an almost simple simply laced Lie group G.

Lusztig-Spaltenstein duality

Expectation There is an order reversing bijection between the sets of symplectic leaves of X and $X^\vee$
(wrong in the case of Langlands dual nilpotent cones).

It extends to a bijection between pairs (a symplectic leaf, an irreducible local system on it).

Thus there is a bijection $\mathcal{F} \mapsto \mathcal{F}^\vee$ between the isomorphism classes of irreducible perverse sheaves on $X, X^\vee$ smooth along symplectic leaves.

Hyperbolic stalks

Furthermore, an integral point $\chi \in \mathfrak{s}_{X,\mathbb{Z}} = H^2(\widetilde{X}, \mathbb{Z})$ defines a partial resolution $\pi_\chi \colon \widetilde{X}_\chi \to X$. On the other hand, χ viewed as an integral point of $\mathfrak{t}_{X^\vee}$ defines a hamiltonian action $\chi \colon \mathbb{C}^\times \curvearrowright X^\vee$. Let us assume that $\widetilde{X}_\chi$ is smooth.

Conjecture (Hikita) There is an isomorphism of rings $H^\bullet(\widetilde{X}_\chi, \mathbb{C}) \simeq \mathbb{C}[(X^\vee)^{\chi(\mathbb{C}^\times)}]$.

Expectation For $\mathcal{F} \in \mathrm{Perv}(X)$, the multiplicity $[\pi_* \mathcal{IC}(\widetilde{X}_\chi) : \mathcal{F}]$ is isomorphic to the hyperbolic stalk $\Phi_\chi \mathcal{F}^\vee$ at the unique $\chi(\mathbb{C}^\times)$-fixed point of $X^\vee$.

Nakajima quiver varieties vs. transversal slices

Returning to our main example
$X = \mathfrak{M}_0(V, W)$, $\widetilde{X} = \mathfrak{M}(V, W)$, $X^\vee = \overline{\mathcal{W}}^\lambda_\mu$, we take the skyscraper sheaf at the vertex of X for $\mathcal{F}$. Then $\mathcal{F}^\vee = \mathcal{IC}(\overline{\mathcal{W}}^\lambda_\mu)$.

We have $V^\lambda_\mu = H_{\mathrm{top}}(\pi^{-1}(0)) = [\pi_* \mathcal{IC}(\widetilde{X}) : \mathcal{F}] = \Phi_\rho \mathcal{IC}(\overline{\mathcal{W}}^\lambda_\mu)$.

- Moreover, the stalk of $\mathcal{IC}(\overline{\mathcal{W}}^\lambda_\mu)$ at the fixed point μ is the associated graded of the hyperbolic stalk (wrt a filtration corresponding to the Brylinski-Kostant filtration of the weight space V^λ_μ).

Higgs branch vs. Coulomb branch

From now on we assume that X is the hamiltonian reduction of a symplectic representation of cotangent type $\mathbf{N} \oplus \mathbf{N}^*$ of a reductive Lie group $\mathcal{G}$ (like in our example with quiver varieties).

Then there is a recipe (Nakajima + Braverman + ...) to construct $X^\vee$ in the framework of 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories:

- X plays the role of the Higgs branch $\mathcal{M}_H(\mathcal{G}, \mathbf{N})$, while $X^\vee$ plays the role of the Coulomb branch $\mathcal{M}_C(\mathcal{G}, \mathbf{N})$.

Variety of triples

Recall: the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}} = \mathcal{G}_{\mathcal{K}}/\mathcal{G}_{\mathcal{O}}$ is the moduli space of pairs $(\mathcal{P}, \sigma)$ where $\mathcal{P}$ is a $\mathcal{G}$-bundle on the formal disc $D = \operatorname{Spec}\mathcal{O}$, and σ is a trivialization of $\mathcal{P}$ on the punctured formal disc $D^* = \operatorname{Spec}\mathcal{K}$.

We need the moduli space $\mathcal{R}_{\mathcal{G},\mathbf{N}}$ of triples $(\mathcal{P}, \sigma, s)$ where s is a section of the associated vector bundle $\mathcal{P}_{\mathrm{triv}} \overset{\mathcal{G}}{\times} \mathbf{N}$ on D^* such that s extends to a regular section of $\mathcal{P}_{\mathrm{triv}} \overset{\mathcal{G}}{\times} \mathbf{N}$ on D, and $\sigma(s)$ extends to a regular section of $\mathcal{P} \overset{\mathcal{G}}{\times} \mathbf{N}$ on D.

Variety of triples

That is, s extends to a regular section of the vector bundle associated to the $\mathcal{G}$-bundle glued from $\mathcal{P}$ and $\mathcal{P}_{\text{triv}}$ on the non-separated scheme glued of 2 copies of D along D^* (raviolo ≎).

The group $\mathcal{G}_{\mathcal{O}}$ acts on $\mathcal{R}_{\mathcal{G},\mathbf{N}}$ by changing the trivialization σ, and we have an evident projection $\mathcal{R}_{\mathcal{G},\mathbf{N}} \to \mathrm{Gr}_{\mathcal{G}}$ forgetting s. The fibers of this projection are profinite dimensional vector spaces: the fiber over the base point is $\mathbf{N}[[z]]$, and all the other fibers are subspaces in $\mathbf{N}[[z]]$ of finite codimension.

One may say that $\mathcal{R}_{\mathcal{G},\mathbf{N}}$ is a $\mathcal{G}_{\mathcal{O}}$-equivariant "constructible profinite dimensional vector bundle" over $\mathrm{Gr}_{\mathcal{G}}$.

Coulomb branch

The $\mathcal{G}_\mathcal{O}$-equivariant Borel-Moore homology $H_\bullet^{\mathcal{G}_\mathcal{O}}(\mathcal{R}_{\mathcal{G},\mathbf{N}})$ is well-defined, and forms an associative algebra with respect to a convolution operation.

This algebra is commutative, finitely generated and integral, and its spectrum $\mathcal{M}_C(\mathcal{G},\mathbf{N}) = \operatorname{Spec} H_\bullet^{\mathcal{G}_\mathcal{O}}(\mathcal{R}_{\mathcal{G},\mathbf{N}})$ is an irreducible normal affine variety of dimension $2\operatorname{rk}(\mathcal{G})$, the *Coulomb branch*.

It is expected to be a (singular) hyper-Kähler manifold.

- It carries a Poisson structure with an open symplectic leaf since $H_\bullet^{\mathcal{G}_\mathcal{O}}(\mathcal{R}_{\mathcal{G},\mathbf{N}})$ can be quantized by considering extra equivariance wrt the loop rotations.

Conjectural slices for Kac-Moody Lie algebras

Conclusion Let $\mathfrak{g}_Q$ be a symmetric Kac-Moody Lie algebra associated to a quiver Q.

Recall: given a dominant weight λ, and another weight $\mu = \lambda - \alpha$, we wanted to construct a slice $\overline{\mathcal{W}}^\lambda_\mu$ in the affine Grassmannian of $\mathfrak{g}_Q$ with IC stalks encoded by Kostka-Viswanath polynomials.

Consider a representation V of Q such that $\alpha = \sum_{i \in Q_0} \dim(V_i)\alpha_i$ with framing W such that $\lambda = \sum_{i \in Q_0} \dim(W_i)\omega_i$.

It gives rise to a representation
$\mathbf{N} = \bigoplus_{e \in Q_1} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i)$ of $\mathcal{G} = GL(V)$.

Generalizations

Conjecture The Coulomb branch $\mathcal{M}_C(\mathcal{G}, \mathbf{N})$ is the desired slice $\overline{\mathcal{W}}^{\lambda}_{\mu}$ in the affine Grassmannian of $\mathfrak{g}_Q$.

It is indeed so if Q is a finite Dynkin quiver, and in a few instances when Q is affine.

Some variations of the above construction (with flavor symmetry) allow to construct the convolution diagrams over slices, and their Beilinson-Drinfeld deformations.