Moduli spaces of local G-shtukas

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Overview

Arithmetic case

- Drinfeld moduli spaces
- Rapoport-Zink: moduli spaces of p-divisible groups of EL/PEL type
  uniformization of Shimura varieties along Newton strata

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- local Shimura varieties (Rapoport-V., Scholze)

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This talk: parallel theory for local function fields

Replace $\mathbb{Q}, \mathbb{Q}_p, \mathbb{Z}_p$ by $\mathbb{F}_p(t), \mathbb{F}_p((t)), \mathbb{F}_p[[t]]$

Loop groups

$\mathbb{F}_q$ with $q = p^r$, $\mathbb{F}$ an algebraic closure

$\mathbf{G}$: parahoric group scheme over $\mathrm{Spec}\, \mathbb{F}_q[\![z]\!]$ with connected reductive special fiber
easiest example: $\mathbf{G} = \mathrm{GL}_n$ or SL_n.

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$L^+\mathbf{G}$: group of positive loops over $\mathbb{F}_q$ associated with $\mathbf{G}$
R an $\mathbb{F}_q$-algebra: $L^+\mathbf{G}(R) = \mathbf{G}(R[\![z]\!])$,

$L\mathbf{G}$: loop group associated with $\mathbf{G}$, i.e. $L\mathbf{G}(R) = \mathbf{G}(R(\!(z)\!))$,

$\mathrm{Flag}_{\mathbf{G}} = L\mathbf{G}/L^+\mathbf{G}$: affine flag variety associated with $\mathbf{G}$.
Pappas-Rapoport: $\mathrm{Flag}_{\mathbf{G}}$ is represented by an ind-projective ind-scheme over $\mathbb{F}_q$.
For GL_n: $\mathrm{Flag}_{\mathbf{G}}(\mathbb{F}) = \{\text{lattices in } \mathbb{F}(\!(z)\!)^n\}$.

Local G-shtukas

Base schemes: $S \in \mathrm{Nilp}_{\mathbb{F}_q[\![\zeta]\!]}$, scheme over $\mathrm{Spec}\ \mathbb{F}_q[\![\zeta]\!]$ with ζ locally nilpotent.

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Local G-shtuka over S: pair $\underline{\mathcal{G}} = (\mathcal{G}, \tau_{\mathcal{G}})$

- $\mathcal{G}$ an $L^+\mathbf{G}$-torsor over S
- $\tau_{\mathcal{G}} : \sigma^* L\mathcal{G} \to L\mathcal{G}$ an isomorphism of the associated $L\mathbf{G}$-torsors.

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Quasi-isogeny: $g : (\mathcal{G}', \tau_{\mathcal{G}'}) \to (\mathcal{G}, \tau_{\mathcal{G}})$: an isomorphism $g : L\mathcal{G}' \to L\mathcal{G}$ with $g \circ \tau_{\mathcal{G}'} = \tau_{\mathcal{G}} \circ \sigma^* g$.

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Example

$\mathbf{G} = \mathrm{GL}_r$: Equivalence (local GL_r-shtukas) $\leftrightarrow$ (shtukas of rank r),
i.e. pairs (M, ϕ_M)

- M a sheaf of $\mathcal{O}_S[\![z]\!]$-modules on S, Zariski-locally free, rank r
- $\phi_M : \sigma^* M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!) \to M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!)$ an isomorphism

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Arithmetic case

Zink's display of a formal p-divisible group over a ring R with p nilpotent

- P a finitely generated projective $W(R)$-module
- $I_R P \subset Q \subset P$ a submodule
- $V^{-1} : Q \to P$ a σ-linear epimorphism such that...

Bounds for local **G**-shtukas - general idea

Want: Bound on the singularities of $\tau_{\mathcal{G}}$

$\rightsquigarrow$ analogy to p-divisible groups

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Let $(\mathcal{G}, \tau_{\mathcal{G}})$ be a local **G**-shtuka over $S \in \mathrm{Nilp}_R$, let S' be an étale covering of S and choose a trivialization $\mathcal{G} \cong (L^+\mathbf{G})_{S'}$. Then $\tau_{\mathcal{G}}$ induces

$$S' \to L\mathbf{G} \widehat{\times}_{\mathbb{F}_q} \mathrm{Spf}\, \mathbb{F}_q[\![\zeta]\!] \to \widehat{\mathrm{Flag}}_{\mathbf{G}} := \mathrm{Flag}_{\mathbf{G}} \widehat{\times}_{\mathbb{F}_q} \mathrm{Spf}\, \mathbb{F}_q[\![\zeta]\!].$$

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Definition

Let $\hat{Z}_R \subset \widehat{\mathrm{Flag}}_{\mathbf{G},R} = \mathrm{Flag}_{\mathbf{G}} \hat{\times}_{\mathbb{F}_q} \mathrm{Spf} R$ a closed ind-subscheme for a finite extension $R/\mathbb{F}_q[\![\zeta]\!]$.
$(\mathcal{G}, \tau_{\mathcal{G}})$ is **bounded** by $\hat{Z}_R$ if for every S' and every trivialization, the map $S' \widehat{\times}_{\mathbb{F}_q[\![\zeta]\!]} \mathrm{Spf} R \to \widehat{\mathrm{Flag}}_{\mathbf{G},R}$ factors through $\hat{Z}_R$.

Bounds for local **G**-shtukas - an example

For $\mathbf{G} = \mathrm{SL}_r$

$$\widehat{\mathrm{Flag}}_{\mathrm{SL}_r}(S) = \Big\{(M, \alpha, \delta) \,\Big|\, M \text{ locally free } \mathcal{O}_S[\![z]\!]\text{-module, rank } r$$

$$\alpha : \wedge^r_{\mathcal{O}_S[\![z]\!]} M \xrightarrow{\sim} \mathcal{O}_S[\![z]\!]$$

$$\delta : M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!) \xrightarrow{\sim} \mathcal{O}_S(\!(z)\!)^{\oplus r}, \text{compatible with } \alpha\Big\}$$

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For $R = \mathbb{F}_q[\![\zeta]\!]$ and $n \in \mathbb{N}$ consider

$$\widehat{\mathrm{Flag}}^{(n)}_{\mathrm{SL}_r}(S) := \left\{(M, \alpha, \delta) \mid \wedge^j_{\mathcal{O}_S[\![z]\!]} \delta(M) \subseteq (z-\zeta)^{n(j^2-jr)} \wedge^j_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!]^{\oplus r} \text{ for } j = 1, \ldots, r\right\}.$$

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- Compatibility with $\alpha \Rightarrow$ equality for $j = r$
- for $j < r$ exponents could be chosen differently
- via representations: can give examples of bounds for other **G**

Bounds for local **G**-shtukas - defining conditions

A **bound** is a closed ind-subscheme $\hat{Z} \subset \mathrm{Flag}_{\mathbf{G}} \hat{\times}_{\mathbb{F}_q} \mathrm{Spf} R$ up to equivalence where

- $R/\mathbb{F}_q[\![\zeta]\!]$ is a finite extension of valuation rings
- $\hat{Z}$ is stable under the left $L^+\mathbf{G}$-action
- $\hat{Z}$ is a ζ-adic formal scheme over $\mathrm{Spf} R$ with quasi-compact special fiber
- There is a faithful representation $\rho : \mathbf{G} \to \mathrm{SL}_r$ over $\mathbb{F}_q[\![z]\!]$ and $n > 0$ such that $\rho_* : \hat{Z} \to \widehat{\mathrm{Flag}}_{\mathrm{SL}_r, R}$ factors through $\widehat{\mathrm{Flag}}^{(n)}_{\mathrm{SL}_r, R}$.
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Let $\breve{R} = \widehat{R^{\mathrm{un}}}$, $E = \mathrm{Quot}\, R$, and $\breve{E} = \mathrm{Quot}\, \breve{R}$.

Moduli spaces for local **G**-shtukas

- $\underline{\mathbb{G}}_0$ a local **G**-shtuka over $\mathbb{F}$
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Theorem (Hartl - V., Arasteh Rad - Hartl)

$\breve{\mathcal{M}}$ is ind-representable by a formal scheme $\breve{\mathcal{M}} = \breve{\mathcal{M}}_{\underline{\mathbb{G}}_0}^{\hat{Z}^{-1}}$ over $\mathrm{Spf}\,\breve{R}$ which is locally formally of finite type and separated.

$\breve{\mathcal{M}}$ is called **Rapoport-Zink space** for bounded local **G**-shtukas.

Moduli spaces for local **G**-shtukas, II

- Direct analog of the definition by Rapoport-Zink for p-divisible groups with EL/PEL-structure
- Here: Also for general groups $\mathbf{G}$, and bounds $\hat{Z}$ not corresponding to a minuscule coweight.

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- Here: Also for general groups $\mathbf{G}$, and bounds $\hat{Z}$ not corresponding to a minuscule coweight.
- Action of $\mathrm{QIsog}_{\mathbb{F}}(\underline{\mathbb{G}}_0)$, the group of self-quasi-isogenies of $\underline{\mathbb{G}}_0$.
  Fix a trivialization $\underline{\mathbb{G}}_0 \cong ((L^+\mathbf{G})_{\mathbb{F}}, b\sigma^*)$ with $b \in L\mathbf{G}(\mathbb{F})$. Then

$$\mathrm{QIsog}_{\mathbb{F}}(\underline{\mathbb{G}}_0) \cong J_b(\mathbb{F}_q(\!(z)\!)) := \{g \in \mathbf{G}(\mathbb{F}(\!(z)\!)) \mid g^{-1}b\sigma(g) = b\}.$$

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- $\breve{\mathcal{M}}^{\mathrm{an}}$: the strictly $\breve{E}$-analytic space associated with $\breve{\mathcal{M}}$.
- $K \subset \mathbf{G}(\mathbb{F}_q(\!(z)\!))$ compact open $\rightsquigarrow$ $\breve{E}$-analytic space $\breve{\mathcal{M}}^K$
  Obtained by trivializing the rational dual Tate module of the universal local **G**-shtuka
  Form a tower: $\breve{\mathcal{M}}^K \to \breve{\mathcal{M}}^{K'}$ for $K \subset K'$

The special fiber

Z^{-1}: the special fiber of $\hat{Z}^{-1}$.

Affine Deligne-Lusztig variety for $b \in L\mathbf{G}(\mathbb{F})$ and Z^{-1}: the reduced closed ind-subscheme $X_{Z^{-1}}(b) \subset \mathrm{Flag}_{\mathbf{G}}$ with

$$X_{Z^{-1}}(b)(\mathbb{F}) = \{g \in \mathrm{Flag}_{\mathbf{G}}(\mathbb{F}) \mid g^{-1} b\sigma(g) \in Z^{-1}(\mathbb{F})\}.$$

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Theorem (Hartl-V.)

The underlying reduced subscheme of $\breve{\mathcal{M}}_{\underline{\mathbb{G}}_0}^{\hat{Z}^{-1}}$ *is* $X_{Z^{-1}}(b)$.

$X_{Z^{-1}}(b)$ is a scheme locally of finite type and separated over $\mathbb{F}$, all of whose irreducible components are projective.

Properties studied by many people (Görtz, He, Kottwitz, Nie, V.,...) $\rightsquigarrow$ He's talk

Relation to moduli of global shtukas

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A **global shtuka** with n legs over a scheme S is a tuple $(\mathscr{G}, s_1, \ldots, s_n, \varphi)$ where

- $\mathscr{G} \in \mathcal{H}^1(C, \mathfrak{G})(S)$ is a $\mathfrak{G}$-torsor over $C \times_{\mathbb{F}_q} S$
- the $s_i \in C(S)$ are pairwise disjoint S-valued points, called the legs
- $\varphi : \sigma^* \mathscr{G}|_{C \times_{\mathbb{F}_q} S \setminus \{\bigcup_i \Gamma_{s_i}\}} \xrightarrow{\sim} \mathscr{G}|_{C \times_{\mathbb{F}_q} S \setminus \{\bigcup_i \Gamma_{s_i}\}}$.

$\nabla_n \mathcal{H}^1(C, \mathfrak{G})$: moduli stack of global $\mathfrak{G}$-shtukas with n legs.

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$\nabla_n \mathcal{H}^1(C, \mathfrak{G})$: moduli stack of global $\mathfrak{G}$-shtukas with n legs.

Fix $c = (c_1, \ldots, c_n) \in C^n(\mathbb{F})$ with $c_i \neq c_j$ for $i \neq j$ and

$\nabla_n \mathcal{H}^1(C, \mathfrak{G})^c$: formal completion of $\nabla_n \mathcal{H}^1(C, \mathfrak{G})$ along c

Relation to moduli of global shtukas II

Then: have function field analogs of

- Serre-Tate's theorem (Arasteh Rad - Hartl)
- Rapoport-Zink uniformization (Arasteh Rad - Hartl, Neupert)
- Product structure of Newton strata into RZ spaces and Igusa varieties (Neupert)

Hodge–Pink structures

Consider

$$\mathbf{G} \times_{\mathbb{F}_q[\![z]\!]} \operatorname{Spec} \mathbb{F}_q(\!(\zeta)\!)[\![z-\zeta]\!]$$

under $z \mapsto z = \zeta + (z - \zeta)$

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Thus base change factors through generic fiber G of $\mathbf{G}$: is reductive.

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$\mathrm{Gr}_G^{B_{dR}}$ over $\operatorname{Spec} \mathbb{F}_q(\!(\zeta)\!)$ with

$$\mathrm{Gr}_G^{B_{dR}} : X \mapsto G(\mathcal{O}_X(\!(z-\zeta)\!))/G(\mathcal{O}_X[\![z-\zeta]\!]),$$

an ind-projective ind-scheme (Pappas-Rapoport, Richarz)

The **space of Hodge-Pink structures** bounded by $\hat{Z}$ is

$$\breve{\mathcal{H}}_{G,\hat{Z}} = \hat{Z}_{\breve{E}} \subseteq \mathrm{Gr}_G^{B_{dR}} \times_{\mathbb{F}_q(\!(\zeta)\!)} \operatorname{Spec} \breve{E}.$$

Hodge–Pink structures II

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Fix a faithful representation $\rho : G_{\mathbb{F}_q((z))} \to \mathrm{GL}_n$, representation space V.
z-isocrystal over $\mathbb{F}$ associated with $\underline{\mathbb{G}}_0 \cong ((L^+\mathbf{G})_{\mathbb{F}}, b\sigma^*)$:

$$\underline{D} = (D, \tau_D) = (V \otimes_{\mathbb{F}_q((z))} \mathbb{F}((z)), \rho(\sigma^* b)\sigma^*)$$

Hodge-Pink structure associated with $\gamma \in \mathrm{Gr}_G^{B_{dR}}(L)$, L an extension of $\breve{E}$:

$$\mathfrak{q}_D(V) = \rho(V) \cdot V \otimes_{\mathbb{F}_q((z))} L[\![z - \zeta]\!] \subset D \otimes_{\mathbb{F}((z))} L((z - \zeta)),$$

a free submodule of full rank.

The period map

Rough idea of the construction: similar as in arithmetic case

- Consider inverse of universal Frobenius $\tau_{\mathcal{G}_{\mathrm{univ}}}$: bounded by $\hat{Z}$
- Use universal quasi-isogeny with $\underline{\mathbb{G}}_{0,\overline{S}} \rightsquigarrow$ Hodge-Pink structure on $\underline{D}$, in $\hat{Z}^{\mathrm{an}}$.

Proposition (Hartl-V.)

This construction induces an étale morphism, the period map

$$\breve{\pi} : \breve{\mathcal{M}}^{\mathrm{an}} \to \breve{\mathcal{H}}^{a}_{G,\hat{Z},b} \subset \breve{\mathcal{H}}^{\mathrm{an}}_{G,\hat{Z}}.$$

Properties of the period map (Hartl-V.)

- Assume that the generic fibre G of $\mathbf{G}$ is unramified, then: image of $\breve{\pi}$ can be described as the locus where two tensor functors coincide.
- arithmetic case: Theorem (Wintenberger) identifying this with $\breve{\mathcal{H}}^{na}_{G,\hat{Z},b}$: 'neutral admissible locus', a union of connected components of $\breve{\mathcal{H}}^{a}_{G,\hat{Z},b}$.
- Natural tensor functor

$$\underline{\mathcal{V}}_b : \mathrm{Rep}_{\mathbb{F}_q((z))}\mathbf{G} \to (\text{local systems of } \mathbb{F}_q\text{-vector spaces on } \breve{\mathcal{H}}^{a}_{G,\hat{Z},b}).$$

- associated tower of coverings of $\breve{\mathcal{H}}^{a}_{G,\hat{Z},b}$ is canonically identified with the tower $\breve{\mathcal{M}}^K$ including Hecke action, $J_b(\mathbb{F})$-action

Thank you!