Ext-analogues of Branching Laws

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Branching Laws

Considering the restriction of representations of a group G to one of its subgroups H has been a very classical subject of much interest both in mathematics and in many applications in physical sciences where it is often referred to as “symmetry breaking” from G to H.

All representations considered in this lecture will be over $\mathbb{C}$.

For example, the famous Clebsch-Gordan theorem about tensor product of representations of $\mathrm{SU}(2)$ which is equivalent to considering restriction of a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ to the diagonal $\mathrm{SU}(2)$ asserts that, Considering the restriction of representations of a group G to one of its subgroups H has been a very classical subject of much interest both in mathematics and in many applications in physical sciences where it is often referred to as "symmetry breaking" from G to H.

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$$\pi_m \otimes \pi_n = \pi_{m+n} + \pi_{m+n-2} + \cdots + \pi_{|m-n|},$$

where π_m is the unique irreducible representation of $\mathrm{SU}(2)$ of dimension $m+1$ for a positive integer $m \geq 0$.

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Two important features of representation theory of compact groups (in particular finite groups) are:

1. finite dimensionality of irreducible representations,
2. Complete reducibility of any finite dimensional representation.

Neither of these will be available to us as we discuss non-compact groups in the rest of the lecture.

An Example

An example to keep in mind when dealing with non-compact groups G is that on the space say $C_c(G)$, the space of compactly supported continuous functions on G, there is a unique G-invariant linear form (up to scaling)

$$C_c(G) \to \mathbb{C},$$

which is

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On the other hand the space $C_c(G)$ does not contain the constant functions as G is non-compact, and therefore one sees that the trivial representation of G appears as a quotient of $C_c(G)$, but not as a submodule of $C_c(G)$.

Considering the restriction of representations of a group G to one of its subgroups H, say of $G = \mathrm{SO}(V)$ to $H = \mathrm{SO}(W)$ where W is a nondegenerate codimension 1 subspace of a finite dimensional quadratic space V over a local field F, has been a very fruitful direction of research especially through its connections to questions on period integrals of automorphic representations:

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thus to L functions.

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- B. Sun and C. Zhu, *Annals of Mathematics*, 2012.

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With the multiplicity one theorem proved, one then goes on to prove a more precise description of the set of irreducible admissible representations π_1 of $\mathrm{SO}(V)$ and π_2 of $\mathrm{SO}(W)$ with

$$\mathrm{Hom}_{\mathrm{SO}(W)}[\pi_1, \pi_2] \neq 0.$$

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- C. Moeglin and J.-L. Waldspurger, *Astérisque*, 2012.
- R. Beuzart-Plessis, *Mém. Soc. Math. Fr.*, 2016.
- R. Beuzart-Plessis, *Compositio Math*, 2015.

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The only classfication known for representations of p-adic groups is what is called Local Langlands correspondence in terms of *enhanced Langlands parameters* which have become available through the works of Jim Arthur, and others, for non-archimedean local fields of characteristic 0 (and due to Langlands for archimedean fields).

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Given the interest in the space

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and in fact homological algebra methods suggest that the simplest answers are not for these individual spaces, but for the alternating sum of their dimensions:

$$\mathrm{EP}[\pi_1, \pi_2] = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2];$$

these hopefully more manageable objects - certainly more flexible - when coupled with vanishing of higher Ext's (when available) may give theorems about

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1. $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ are finite dimensional,
2. $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2] = 0$ for i large.

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This is a standard application of the projective resolution of the trivial representation $\mathbb{C}$ of G provided by the (Bruhat-Tits) building associated to G.

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The abelian category $\mathcal{R}(G)$ has enough projectives and enough injectives, e.g. for any compact open subgroup K of G, $\mathrm{ind}_K^G(\mathbb{C})$ is a projective object in $\mathcal{R}(G)$, and $\mathrm{Ind}_K^G(\mathbb{C})$ is an injective object in $\mathcal{R}(G)$ (we use ind for compactly supported induction and Ind for induction without compact support condition);

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Since $\mathcal{R}(G)$ has enough projectives and enough injectives, it is meaningful to talk about $\mathrm{Ext}^i_G[\pi_1, \pi_2]$ as the derived functors of $\mathrm{Hom}_G[\pi_1, \pi_2]$.

Preliminaries

Irreducible representations of a p-adic group G are understood in 2 steps:

1. Supercuspidal representations: These are irreducible representations of G which do not arise as sub-quotients of principal series representations $\mathrm{Ind}_P^G(\lambda)$ for an irreducible representation λ of M treated as a representation of a proper parabolic subgroup $P = MN$.
2. Irreducible representations π arising as subquotients of principal series representations $\mathrm{Ind}_P^G(\lambda)$ with λ, a supercuspidal representation of a proper Levi subgroup M of the parabolic P; the data (M, λ) is unique up to conjugacy, called the cuspidal support of the representation π of G.

Finite Length Representations

The following theorem summarizes some key properties of the Euler-Poincaré pairing, the last part known as Kazhdan orthogonality, known only in characteristic zero.

Theorem

Let π and π' be finite-length, smooth representations of a reductive p-adic group G. Then:

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4. $\mathrm{EP}_G[\pi, \pi'] = \int_{C_{ellip}} \Theta(c)\bar{\Theta}'(c)\, dc$, *where Θ and Θ' are the characters of π and π' which are assumed to have the same unitary central character, and dc is a natural measure on the set C_{ellip} of regular elliptic conjugacy classes in G.*

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is an exact sequence of smooth G-modules, and if any two of

$$\mathrm{EP}_G[\pi_1, \pi'], \mathrm{EP}_G[\pi, \pi'], \mathrm{EP}_G[\pi_2, \pi'],$$

make sense, then so does the third (finite dimensionality of the Ext groups, and zero beyond a stage), and

$$\mathrm{EP}_G[\pi, \pi'] = \mathrm{EP}_G[\pi_1, \pi'] + \mathrm{EP}_G[\pi_2, \pi'].$$

Finite dimensionality

For the proof of the finite dimensionality of Ext^i we note that unlike $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, where we will have no idea how to prove finite dimensionality if both π_1 and π_2 are cuspidal,

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The finite dimensionality of $\mathrm{Ext}^i[\pi_1, \pi_2]$ when one of the representations π_1, π_2 is a full principal series representation, is achieved by an inductive argument both on n and on the split rank of the Levi from which the principal series arises.

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General Finiteness Theorems

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However, the approach via Bessel models which intervene when analyzing principal series representations of $\mathrm{SO}_{n+1}(F)$ when restricted to $\mathrm{SO}_n(F)$ has as a bonus, explicit answers about Euler-Poincaré characteristics (at least in some cases).

Adjoint Functors

Several assertions about Hom spaces can be converted into assertions about Ext^i. The following generality allows one to do so.

Proposition

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and $\mathcal{F}$ a functor from $\mathcal{A}$ to $\mathcal{B}$, and $\mathcal{G}$ a functor from $\mathcal{B}$ to $\mathcal{A}$. Assume that $\mathcal{G}$ is a left adjoint of $\mathcal{F}$, i.e., there is a natural equivalence of functors:

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Several assertions about Hom spaces can be converted into assertions about Ext^i. The following generality allows one to do so.

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Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and $\mathcal{F}$ a functor from $\mathcal{A}$ to $\mathcal{B}$, and $\mathcal{G}$ a functor from $\mathcal{B}$ to $\mathcal{A}$. Assume that $\mathcal{G}$ is a left adjoint of $\mathcal{F}$, i.e., there is a natural equivalence of functors:

$$\mathrm{Hom}_{\mathcal{B}}[X, \mathcal{F}(Y)] \cong \mathrm{Hom}_{\mathcal{A}}[\mathcal{G}(X), Y].$$

Then,

1. *If $\mathcal{F}$ and $\mathcal{G}$ are exact functors, then $\mathcal{F}$ maps injective objects of $\mathcal{A}$ to injective objects of $\mathcal{B}$, and $\mathcal{G}$ maps projective objects of $\mathcal{B}$ to projective objects of $\mathcal{A}$.*
2. *If $\mathcal{F}$ and $\mathcal{G}$ are exact functors, then*

$$\mathrm{Ext}^i_{\mathcal{B}}[X, \mathcal{F}(Y)] \cong \mathrm{Ext}^i_{\mathcal{A}}[\mathcal{G}(X), Y].$$

Frobenius Reciprocity for Ext Groups

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2. *For any smooth representation U of H, and V of G,*

$$\mathrm{Ext}^i_G[V, \mathrm{Ind}_H^G U] \cong \mathrm{Ext}^i_H[V, U],$$

$$\mathrm{Ext}^i_H[V, U^\vee] \cong \mathrm{Ext}^i_G[V, \mathrm{Ind}_H^G(U^\vee)] \cong \mathrm{Ext}^i_G[\mathrm{ind}_H^G U, V^\vee].$$

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$$\mathrm{Ext}^i_G[V, \mathrm{Ind}_P^G U] \cong \mathrm{Ext}^i_M[V_N, U],$$

$$\mathrm{Ext}^i_G[\mathrm{Ind}_P^G U, V] \cong \mathrm{Ext}^i_M[U, V_{N^-}].$$

We begin by recalling the following basic result in this context.

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Theorem

Given an irreducible generic representation π_1 *of* $\mathrm{GL}_{n+1}(F)$, *and an irreducible generic representation* π_2 *of* $\mathrm{GL}_n(F)$,

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The following theorem can be considered as the Euler-Poincaré version of the above theorem.

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Let π_1 be an admissible representation of $\mathrm{GL}_{n+1}(F)$ of finite length, and π_2 an admissible representation of $\mathrm{GL}_n(F)$ of finite length. Then, $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$ are finite dimensional vector spaces over $\mathbb{C}$, and

$$\mathrm{EP}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = \dim \mathrm{Wh}(\pi_1) \cdot \dim \mathrm{Wh}(\pi_2),$$

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$$\mathrm{EP}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = \dim \mathrm{Wh}(\pi_1) \cdot \dim \mathrm{Wh}(\pi_2),$$

where $\mathrm{Wh}(\pi_1)$, resp. $\mathrm{Wh}(\pi_2)$, denotes the space of Whittaker models for π_1, resp. π_2, with respect to fixed non-degenerate characters on the maximal unipotent subgroups in $\mathrm{GL}_{n+1}(F)$ and $\mathrm{GL}_n(F)$.

On the Proof of the Theorem on EP

The proof of this theorem is accomplished using some results of Bernstein and Zelevinsky regarding the structure of representations of $\mathrm{GL}_{n+1}(F)$ restricted to the mirabolic subgroup.

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For a representation π of $\mathrm{GL}_{n+1}(F)$, Bernstein-Zelevinsky define

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Of crucial importance is the fact that if π is of finite length for $\mathrm{GL}_{n+1}(F)$, then π^i are representations of finite length of $\mathrm{GL}_{n+1-i}(F)$.

On the Proof of the Theorem on EP: Continued

Bernstein-Zelevinsky prove that the restriction of an admissible representation π of $GL_{n+1}(F)$ to the mirabolic E_n has a finite filtration whose successive quotients are described by the derivatives π^i of π.

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Lemma

If V and W are any two finite length representations of $\mathrm{GL}_d(F)$, then if $d > 0$,

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If $d = 0$, then of course

$$EP[V, W] = \dim V \cdot \dim W.$$

Vanishing of Ext Groups for Generic Representations

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for all $i > 0$.

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This conjecture has recently been proved:

- K. Y. Chan and G. Savin, *A vanishing Ext-branching theorem for* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$, 2018. arXiv:1803.09131.

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In particular, in general a generic representation of $\mathrm{GL}_{n+1}(F)$ when restricted to $\mathrm{GL}_n(F)$ is not a projective representation in $\mathcal{R}(\mathrm{GL}_n(F))$ although this is the case for supercuspidal representations of $\mathrm{GL}_{n+1}(F)$. For example, even for generic π_1, there are sometimes nongeneric π_2 with $Ext^i(\pi_1, \pi_2) \neq 0$, $i > 0$.

The paper of Chan and Savin quoted above proves that any discrete series representation π of $\mathrm{GL}_{n+1}(F)$ when restricted to $\mathrm{GL}_n(F)$ is a projective representation in $\mathcal{R}(\mathrm{GL}_n(F))$, The paper of Chan and Savin quoted above proves that any discrete series representation π of $\mathrm{GL}_{n+1}(F)$ when restricted to $\mathrm{GL}_n(F)$ is a projective representation in $\mathcal{R}(\mathrm{GL}_n(F))$, therefore

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As a consequence, it will follow from the duality theorem of Schneider-Stuhler below, that discrete series representations of $\mathrm{GL}_{n+1}(F)$ contain no irreducible submodule of $\mathrm{GL}_n(F)$.

In the next few sections we will discuss Euler-Poincaré characteristic for branching laws for classical groups, restricting ourselves to the case of $G = SO_{n+1}(F)$ and $H = \mathrm{SO}_n(F)$.

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We will use Bessel subgroups, and Bessel models without defining them, except to recall that these are defined whenever we have quadratic spaces $W \subset V$, with V/W an odd dimensional split quadratic space. The Bessel subgroup $\mathrm{Bes}(V, W)$ is a subgroup of $\mathrm{SO}(V)$ of the form $\mathrm{SO}(W) \cdot U$ where U is a unipotent subgroup of $\mathrm{SO}(V)$ which comes with a character normalized by $\mathrm{SO}(W)$.

Classical Groups: A Result about Principal Series

Proposition

For a principal series representation $\pi = \pi_0 \rtimes \sigma_0$ *of* $\mathrm{SO}(V)$ *where* σ_0 *is an admissible representation of* $\mathrm{SO}(W)$*, and* π' *is an admissible representation of* $\mathrm{SO}(V')$ *where*

$$W \subset V' \subset V$$

with V' *a non-degenerate codimension 1 subspace of the quadratic space* V *with*

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we have

$$\mathrm{EP}_{\mathrm{SO}(V')}[\pi, \pi'] = \mathrm{EP}_{\mathrm{Bes}(V',W)}[\pi', \sigma_0] \cdot \dim \mathrm{Wh}(\pi_0).$$

Full Principal Series

Definition: A finite length representation π of a classical group will be called a *full principal series* if it is irreducible and supercuspidal, or is of the form $\pi = \pi_0 \rtimes \sigma_0$ with both π_0 and σ_0 irreducible, and σ_0 supercuspidal.

The following result is a consequence of the previous proposition together with the fact that if σ is a cuspidal representation of $\mathrm{SO}(W)$, then $\sigma \otimes \psi$ is an injective module for $\mathrm{Bes}(V, W)$.

Calculation of EP for Classical Groups

The following result is a consequence of the previous proposition together with the fact that if σ is a cuspidal representation of $\mathrm{SO}(W)$, then $\sigma \otimes \psi$ is an injective module for $\mathrm{Bes}(V, W)$.

Proposition

Let π be a finite length representation of $\mathrm{SO}(V)$, and π' of $\mathrm{SO}(V')$ where $V' \subset V$ is a non-degenerate codimension 1 subspace of the quadratic space V. Assume that π is a full principal series, and π' is an irreducible representation of $\mathrm{SO}(V')$. Then,

$$\mathrm{EP}_{\mathrm{SO}(V')}[\pi, \pi'] \in \{0, 1\}.$$

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$$\mathrm{EP}_{\mathrm{SO}(V')}[\pi, \pi'] \in \{0, 1\}.$$

If $\pi = \pi_0 \rtimes \sigma_0$ of $\mathrm{SO}(V)$ where σ_0 is an admissible representation of $\mathrm{SO}(W)$ with $\dim W \leq 1$,

$$\mathrm{EP}_{\mathrm{SO}(V')}[\pi, \pi'] = \dim \mathrm{Wh}(\pi) \cdot \dim \mathrm{Wh}(\pi')$$

(if $\dim W = 1$, $\mathrm{Wh}(\pi')$ is for a particular character of a maximal unipotent subgroup of $\mathrm{SO}(V')$).

In the previous slide, we see a large number of cases when the Euler-Poincaré characteristic is 0 or 1. Is there a multiplicity one result for EP, or for Ext^i?

An Integral Formula of Waldspurger

Theorem

Let $V = X + D + W + Y$ be a quadratic space over the non-archimedean local field F with W a quadratic subspace of codimension $2k+1$ as above. Then for any irreducible admissible representation σ of $\mathrm{SO}(V)$ and irreducible admissible representation σ' of $\mathrm{SO}(W)$,

$$c(\sigma, \sigma') := \sum_{T \in \mathcal{T}} |W(H, T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) \Delta^k(t) dt,$$

is a finite sum of absolutely convergent integrals. (The Haar measure on $T(F)$ is normalized to have volume 1.)

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is a finite sum of absolutely convergent integrals. (The Haar measure on $T(F)$ is normalized to have volume 1.) If either σ is a supercuspidal representation of $\mathrm{SO}(V)$, and σ' is arbitrary irreducible admissible representation of $\mathrm{SO}(W)$, or both σ and σ' are tempered representations, then

$$c(\sigma, \sigma') = \dim \mathrm{Hom}_{\mathrm{Bes}(V,W)}[\sigma, \sigma'].$$

Conjectured EP Formula

Given the theorem of Waldspurger, it is most natural to propose the following conjecture on Euler-Poincaré pairing following the earlier notation of $V = X + D + W + Y$, a quadratic space over the non-archimedean local field F with W a quadratic subspace of V of codimension $2k+1$. For a finite length representation σ of $\mathrm{SO}(V)$ and σ' of $\mathrm{SO}(W)$, we have:

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Conjecture

1. $\mathrm{EP}_{\mathrm{Bes}(V,W)}[\sigma, \sigma'] = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\sigma, \sigma']$
$= \sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) \Delta^k(t) dt.$

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$= \sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) \Delta^k(t) dt.$
2. *If σ and σ' are irreducible tempered representations, then*
$$\mathrm{Ext}^i_{\mathrm{Bes}(V,W)}[\sigma, \sigma'] = 0$$
for $i > 0$.

Remark on the Conjectured EP Formula

- Waldspurger's theorem is equivalent to the conjectural statement on Euler-Poincaré characteristic if σ or σ' is supercuspidal (but not proved if σ' is supercuspidal).

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Remark on the Conjectured EP Formula

- Waldspurger's theorem is equivalent to the conjectural statement on Euler-Poincaré characteristic if σ or σ' is supercuspidal (but not proved if σ' is supercuspidal).
- Part 2 of the conjecture is there as the simplest possible explanation of Waldspurger's theorem for tempered representations!
- Waldspurger integral formula is available also in the work of R. Beuzart-Plessis for unitary groups. It appears to be an important problem to find a general integral formula for EP for spherical varieties $G \times H/\Delta(H)$.

An Example for the Conjectured EP Formula

Assume that either $G = \mathrm{SO}_{n+1}(F)$ is a split group, and σ is induced from a character of a Borel subgroup of G, or $H = \mathrm{SO}_n(F)$ is a split group and σ' is induced from a character of a Borel subgroup of H. Then the conjectured formula on Euler-Poincaré becomes

$$\mathrm{EP}[\sigma, \sigma'] = 1$$

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Assume that either $G = \mathrm{SO}_{n+1}(F)$ is a split group, and σ is induced from a character of a Borel subgroup of G, or $H = \mathrm{SO}_n(F)$ is a split group and σ' is induced from a character of a Borel subgroup of H. Then the conjectured formula on Euler-Poincaré becomes

$$\mathrm{EP}[\sigma, \sigma'] = 1$$

which is a consequence of one of the earlier propositions since in that case there are no elliptic elements in H except the trivial which is contained in a split torus inside the corresponding Borel subgroup.

Analogy with Riemann-Roch Theorem

We consider the *Waldspurger integral formula* as some kind of a *Riemann-Roch theorem.*

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$$\begin{aligned} EP(X,\mathfrak{F}) &= \sum_i (-1)^i \dim H^i(X,\mathfrak{F}) \\ &= \sum_i (-1)^i \dim \mathrm{Ext}^i(\mathrm{O}_X,\mathfrak{F}) \\ &= \int_X (T_X \cdot c(\mathfrak{F})). \end{aligned}$$

Analogy with Riemann-Roch Theorem

In our case,

$$EP[\pi_1, \pi_2] = \sum_i (-1)^i \dim \mathrm{Ext}^i_H[\pi_1, \pi_2]$$

is conjecturally expressed as

Analogy with Riemann-Roch Theorem

In our case,

$$EP[\pi_1, \pi_2] = \sum_i (-1)^i \dim \mathrm{Ext}^i_H[\pi_1, \pi_2]$$

is conjecturally expressed as

$$EP[\pi_1, \pi_2] = \int_X T_X \cdot c(\pi_1, \pi_2),$$

where X is a certain set of elliptic tori in H, T_X is a function on this set of elliptic tori, and $c(\pi_1, \pi_2)$ is a function on these elliptic tori defined in terms of the germ expansion of π_1 and π_2.

Existence of Submodules

In the archimedean case, several papers of T. Kobayashi study the restriction problem for $(\mathfrak{g}, K)$-modules in the sense of sub-modules, see for example the paper:

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Many earlier works suggest that $\mathrm{Hom}_H[\pi_2, \pi_1] = 0$ whenever π_1 is an irreducible tempered representation of G (assumed to be simple) unless H has compact center, and π_2 is a supercuspidal representation of it.

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This is not quite correct, and in the next sections we will see how to answer this.

The Schneider-Stuhler Duality Theorem

The following theorem is a mild generalization of a duality theorem of Schneider and Stuhler with Nori; it turns questions on $\mathrm{Ext}^i[\pi_1, \pi_2]$ to $\mathrm{Ext}^j[\pi_2, \pi_1]$, and is of considerable importance to our theme.

The Schneider-Stuhler Duality Theorem

Theorem

Let G be a reductive p-adic group, and π an irreducible admissible representation of G. Let $d(\pi)$ be the largest integer $i \geq 0$ such that there is an irreducible admissible representation π' of G with $\mathrm{Ext}^i_G[\pi, \pi'] \neq 0$.

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Let G be a reductive p-adic group, and π an irreducible admissible representation of G. Let $d(\pi)$ be the largest integer $i \geq 0$ such that there is an irreducible admissible representation π' of G with $\mathrm{Ext}^i_G[\pi, \pi'] \neq 0$.

1. *There is a unique irreducible representation π' of G with $\mathrm{Ext}^{d(\pi)}_G[\pi, \pi'] \neq 0$.*
2. *The representation π' in (1) is nothing but $D(\pi)$ where $D(\pi)$ is the Aubert-Zelevinsky involution of π, and $d(\pi)$ is the split rank of the center of the Levi subgroup M of G which carries the cuspidal support of π.*
3. $\mathrm{Ext}^{d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C}$.
4. *For any smooth representation π' of G, the bilinear pairing*

$$(*) \quad \mathrm{Ext}^i_G[\pi, \pi'] \times \mathrm{Ext}^j_G[\pi', D(\pi)] \to \mathrm{Ext}^{i+j=d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C},$$

is non-degenerate.

The Schneider-Stuhler Duality Theorem

- P. Schneider and U. Stuhler, *Inst. Hautes Études Sci. Publ. Math.*, 1997.
- M. Nori and D. Prasad, 2017. arXiv:1711.01908.

An Application of the Duality Theorem

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The following proposition gives a complete classification of irreducible submodules π of the tensor product $\pi_1 \otimes \pi_2$ of two (irreducible, infinite dimensional) representations π_1, π_2 of $\mathrm{GL}_2(F)$ with the product of their central characters trivial.

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In this case, the required vanishing of $\mathrm{Ext}^1_{PGL_2(F)}[\pi_1 \otimes \pi_2, \pi_3]$ follows because

$$\mathrm{Ext}^2_{PGL_2(F)}[\pi_1 \otimes \pi_2, \pi_3] = 0$$

as the rank of $PGL_2(F)$ is 1, and both EP and Hom spaces have the same dimension.

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Proposition

Let π_1, π_2 be two irreducible admissible infinite dimensional representations of $\mathrm{GL}_2(F)$ with product of their central characters trivial. Then the following is a complete list of irreducible sub-representations π of $\pi_1 \otimes \pi_2$ as $\mathrm{PGL}_2(F)$-modules.

1. *π is a supercuspidal representation of $\mathrm{PGL}_2(F)$, and appears as a quotient of $\pi_1 \otimes \pi_2$.*

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1. *π is a supercuspidal representation of $\mathrm{PGL}_2(F)$, and appears as a quotient of $\pi_1 \otimes \pi_2$.*
2. *π is a twist of the Steinberg representation, which we assume by absorbing the twist in π_1 or π_2 to be the Steinberg representation St of $\mathrm{PGL}_2(F)$. Then St is a submodule of $\pi_1 \otimes \pi_2$ if and only if π_1, π_2 are both irreducible principal series representations, and $\pi_1 \cong \pi_2^\vee$.*

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In fact $\mathrm{Ext}^i_H[\pi_2, \pi_1]$ is zero beyond the split rank of the center of this Levi by generalities, so $\mathrm{Ext}^i_H[\pi_1, \pi_2]$ is non-zero only for $i = 0$, whereas $\mathrm{Ext}^i_H[\pi_2, \pi_1]$ is non-zero only for the largest possible i.

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6. Special role played by $X = \mathbb{P}^d(F)$ (of course we have our own, *her all-embracing majesty*, $\mathrm{GL}_n(F)$).

Thank You!