Asymptotic invariants of locally symmetric spaces

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E.g. about the topology and the geometry of M, or group theoretic and arithmetic properties of Γ.

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One can still ask more refined questions, especially when focusing on arithmetic surfaces.

Betti numbers

Theorem (Gromov, 1985)

Given d there is $C = C(d)$ such that for every analytic Hadamard d-manifold M of sectional curvature $-1 \leq K \leq 0$, we have

$$\sum_{i=1}^{d} b_i(M) \leq C vol(M).$$

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The proof occupies half of Ballmann–Gromov–Schroeder book 'Manifolds of Nonpositive Curvature'.

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Corollary (Kazhdan–Margulis theorem)

There is a uniform lower bound on the co-volume of lattices:

$$vol(G/\Gamma) \geq \frac{1}{C}.$$

Arithmetic group

Definition (Arithmetic subgroup)

Let G be a real Lie group and $\Gamma \leq G$ a discrete subgroup. Then Γ is called an arithmetic subgroup of G if

- there exists a $\mathbb{Q}$ algebraic group $\mathbb{H}$ and
- surjective map $f : \mathbb{H}(\mathbb{R}) \to G$ with compact kernel, such that
- Γ is commensurable with $f(\mathbb{H}(\mathbb{Z}))$.

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Theorem (Corlette, Gromov–Schoen, 92)

Lattices in the rank one groups $Sp(n, 1)$ and F_4^{-20} are arithmetic.

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Conjecture (G, 2003)

Let $X = G/K$ *be a symmetric space of non-compact type. Then there are constants* α, D *such that every arithmetic* X*-manifold* $M = \Gamma \backslash X$ *is homotopy equivalent to some* $(D, \alpha \cdot \mathrm{vol}(M))$*-simplicial complex.*

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Remark (Conjecture)

The arithmeticity assumption is needed only for the case $G = SL_2(\mathbb{C})$.

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Let $\Gamma \leq G$ *be an anisotropic (i.e. of* $\mathbb{Q}$*-rank* 0*) torsion free arithmetic group. Then the Mahler measure of the characteristic polynomial of every non-trivial element in* Γ *is at least* 0.1*.*

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If the Margulis conjecture is true then compact arithmetic manifolds are uniformly thick.

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Thus, the conjecture holds for the class of ϵ-thick manifolds.

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Suppose that G is neither locally isometric to $SL_2(\mathbb{C})$, $SL_3(\mathbb{R})$ nor to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, then there are constants D, α such that any torsion free lattice $\Gamma \leq G$ is isomorphic to the fundamental group of some $(D, \alpha \cdot vol(\Gamma \backslash G))$-simplicial complex.

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- I.e. any X-manifold M admits a $(D, \alpha \text{vol}(M))$-simplicial complex with the same fundamental group.
- This result is sufficient for many applications.
- Note that by Mostow rigidity theorem $\Gamma = \pi_1(M)$ determines M.

The result of Fraczyk

Recently Mikolaj Fraczyk proved the conjecture for $\mathrm{SL}_2(\mathbb{C})$.

Theorem (Fraczyk, 2017)

There are constants α, D such that every arithmetic hyperbolic 3-manifold M is homotopy equivalent to a $(D, \alpha \cdot \mathrm{vol}(M))$-simplicial complex.

Corollary 1: Effective presentation of π_1

Exercise

The fundamental group of a simplicial complex $\mathcal{R}$ *admits a presentation* $\langle \Sigma : R \rangle$ *with*

- $|\Sigma| \leq$ *the number of edges in* $\mathcal{R}^1$,
- $|R| \leq$ *the number of 2-cells in* $\mathcal{R}^2$ *and*
- *each relation in* R *is of length* ≤ 3.

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Thus, we obtained:

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Let G be a connected semisimple Lie group without compact factors $\not\cong SL_2(\mathbb{C}),\ SL_3(\mathbb{R}),\ SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Then there is a constant $C = C(G)$ such that every torsion free irreducible lattice $\Gamma \leq G$ admits a presentation $\Gamma \cong \langle \Sigma : R \rangle$ with $|\Sigma|, |R| \leq C \cdot vol(\Gamma\backslash G)$, such that every relation $w \in R$ is of length at most 3.

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Theorem

Let G be a connected semisimple Lie group without compact factors $\not\cong SL_3(\mathbb{R}),\ SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Then there is a constant $C = C(G)$ such that every torsion free irreducible arithmetic lattice $\Gamma \le G$ admits a presentation $\Gamma \cong \langle \Sigma : R \rangle$ with $|\Sigma|, |R| \le C \cdot vol(\Gamma \backslash G)$, such that every relation $w \in R$ is of length at most 3.

Corollary 2: Effective Wang's theorem

Theorem (Wang 1972)

Let G be a semisimple Lie group without compact factors, not locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$. Then for any $v > 0$, G admits only finitely many conjugacy classes of irreducible lattices $\Gamma \leq G$ with $vol(\Gamma\backslash G) \leq v$.

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Theorem (G, 2004 & 2011)

Suppose that $\dim(X) \geq 4$ *then* $\rho_X(v) \leq v^{cv}$ *for an appropriate constant* $c = c(G)$.

Counting Hyperbolic Manifolds

Theorem (Burger, G, Lubotzky, Mozes 2002)

For every $d \geq 4$ there are constants a, b such that for all v sufficiently large

$$v^{av} \leq \rho_{\mathbb{H}^d}(v) \leq v^{bv}.$$

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For $G = PSL_2(\mathbb{R})$ we obtained the following precise result:

Theorem (Counting arithmetic surfaces [BGLM 2010])

For $H = PSL(2, \mathbb{R})$ we have

$$\lim_{x \to \infty} \frac{\log \mathrm{AL}_H(x)}{x \log x} = \frac{1}{2\pi}.$$

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If we restrict the counting to arithmetic manifolds we obtain (almost) polynomial growth.

Corollary

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Counting lattices up to quasi ismetry

Let $\mathcal{QI}_d(v)$ be the number of torsion free lattices in $PO(d,1) \cong \mathrm{Isom}(\mathbb{H}^d)$ of co-volume $\leq v$ considered up to quasi isometry.

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And again, the number of arithmetic lattices up to QI is polynomial in the volume.

Counting higher rank manifolds

Theorem (Belolipetsky, Lubotzky 2005)

Let H be a simple Lie group of real rank at least 2. Then

1. *There exists a positive constant a such that $\rho_H(v) \geq v^{a \log v}$.*
2. *Assuming the CSP and Margulis–Platonov conjecture, there exists a positive constant b such that $\rho_H(v) \leq v^{b \log v}$ for all sufficiently large v.*

Back to cohomology — bounding the torsion

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Theorem (Bader, G, Sauer 2018)

For every $d \neq 3$, there exists $C = C_d > 0$ such that for every complete d-dimensional Riemannian manifold M of normalized bounded negative curvature and for every degree k,

$$\log |torsH_k(M;\mathbb{Z})| \leq C vol(M).$$

Dimension 3

Theorem

There is a family $(M_{p,q})_{(p,q)\in\mathbb{N}^2}$ *of pairwise non-homotopy equivalent closed hyperbolic 3-manifolds* $M_{p,q}$ *satisfying*

- $vol(M_{p,q}) < 2.03$,
- $H_1(M_{p,q}) \cong \mathbb{Z}/p\mathbb{Z}$.

Benjamini–Schramm convergence, Farber property

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For $r > 0$, the r-thin part of a Riemannian manifold M is

$$M_{\leq r} = \{x \in M : \ \text{InjRad}_M(x) \leq r\}.$$

Lemma

M_n is Farber iff for every $r > 0$, $\frac{vol((M_n)_{\leq r})}{vol(M_n)} \to 0$.

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Claim

A sequence of IRS μ_n is Farber iff it converges to the trivial IRS $\delta_{\{1\}}$.

Higher Rank manifolds are fat

Theorem (Abert, Bergeron, Biringer, G, Nikolov, Raimbault, Samet)

Suppose that G has Kazhdan's property (T) and rank ≥ 2. Then the sequence (M_n) of all irreducible finite volume X-manifolds, is Farber.

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Suppose that G has Kazhdan's property (T) and rank ≥ 2. Then the sequence (M_n) of all irreducible finite volume X-manifolds, is Farber. Moreover, $\delta_{\{1\}}$ is the only accumulation point of the compact space EIRS(G) of ergodic IRS of G.

Refined asymptotic of Betti Numbers

Let $X = G/K$ be a symmetric space of non-compact type.

Theorem (Abert, Bergeron, Biringer, G, Nikolov, Raimbault, Samet)

Let (M_n) be a Farber sequence of X-manifolds. Then

$$\lim_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k(X)$$

for $0 \leq k \leq \dim(X)$.

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Let $X = G/K$ be a symmetric space of non-compact type.

Theorem (Abert, Bergeron, Biringer, G, Nikolov, Raimbault, Samet)

Let (M_n) be a Farber sequence of X-manifolds. Then

$$\lim_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k(X)$$

for $0 \le k \le \dim(X)$.

The L^2-Betti numbers of X satisfy:

$$\beta_k(X) = \begin{cases} 0, & k \neq \frac{1}{2}\dim X \\ \frac{\chi(X^*)}{\mathrm{vol}(X^*)}, & k = \frac{1}{2}\dim X. \end{cases}$$

Note also that $\chi(X^*) = 0$ unless $\mathrm{rank}_{\mathbb{C}}(G) = \mathrm{rank}_{\mathbb{C}}(K)$.

Convergence of Plancherel measures

Theorem (7s)

Let (Γ_n) be a uniformly discrete Farber sequence of (uniform) lattices in G. Then for any relatively compact ν_G-regular open subset $S \subset \hat{G}$ we have

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Here ν^G denotes the Plancherel measure of the right regular representation on $L^2(G)$.

For a uniform lattice $\Gamma \leq G$ the relative Plancherel measure associated with $L_2(\Gamma \backslash G)$ is given by

$$\nu_\Gamma = \frac{1}{\mathrm{Vol}(G/\Gamma)} \sum_{\pi \in \hat{G}} m(\pi, \Gamma) \delta_\pi$$

where $m(\pi, \Gamma)$ is the multiplicity of π in $L_2(\Gamma \backslash G)$.

Convergence of Plancherel measures

For $\pi \in \hat{G}$ let $d(\pi)$ denote the *formal degree* of π in the regular representation. Thus $d(\pi) = 0$ unless π is a discrete series representation.

Corollary (7s)

Let (Γ_n) be a uniformly discrete Farber sequence of lattices in G. Then for all $\pi \in \hat{G}$, we have

$$\frac{m(\pi, \Gamma)}{vol(\Gamma \backslash G)} \to d(\pi).$$

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