Representation theory of W-algebras and Higgs branch conjecture

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- W-algebras appeared in '80s in physics in the study of the two-dimensional conformal field theories.
- W-algebras are closely connected with integrable systems, (quantum) geometric Langlands program (e.g. [T.A.-Frenkel '18]), four-dimensional gauge theory ([Alday-Gaiotto-Tachikawa '10]), etc.

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relations: $[\mathbf{c}, W_3] = 0$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c}$,

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$$= (m-n)\left(\tfrac{1}{15}(m+n+3)(m+n+2) - \tfrac{1}{6}(m+2)(n+2)\right) L_{m+n}$$
$$+\tfrac{16}{22+5\mathbf{c}}(m-n)\Lambda_{m+n} + \tfrac{1}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}\mathbf{c},$$

where $\Lambda_n = \sum_{k\geq 0} L_{n-k}L_k + \sum_{k<0} L_k L_{n-k} - \frac{3}{10}(n+2)(n+3)L_n$.

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W-algebras are not Lie algebras in general but vertex algebras.

Representations of W_3-algebra

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$$L_n v = W_n v = 0 \ (n > 0),$$
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For a highest weight representation M of W_3 the (normalized) character

$$\chi_M(q) = \mathrm{tr}_M(q^{L_0 - \frac{\mathbf{c}}{24}})$$

makes sense.

Quantized Drinfeld-Sokolov reduction

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$V^k(\mathfrak{g})$: the universal affine vertex algebra associated with $\mathfrak{g}$ at level k (vertex algebra associated with the affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$).

Examples of $\mathcal{W}^k(\mathfrak{g}, f)$

1). $\mathcal{W}^k(\mathfrak{g}, 0) = V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]+\mathbb{C}K)} \mathbb{C}_k$
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2). $\mathcal{W}^k(\mathfrak{sl}_2, f_{prin}) =$ the Virasoro vertex algebra of central charge $1 - 6(k+1)^2/(k+2)$ (if k is not critical, i.e., $k \neq -2$).

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5). Almost all superconformal algebras are realized as the W-algebra $\mathcal{W}^k(\mathfrak{g}, f_{min})$ associated with some Lie superalgebra $\mathfrak{g}$ and a minimal nilpotent element f_{min} ([Kac-Roan-Wakimoto '03]).

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Presentation of $\mathcal{W}^k(\mathfrak{g}, f)$ by generators and relations are not known in general.

Drinfeld-Sokolov reduction functor

The definition of $\mathcal{W}^k(\mathfrak{g}, f)$ by the quantized Drinfeld-Sokolov reduction gives rise to a functor

$$V^k(\mathfrak{g})\text{-Mod} \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod},$$
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Remark

The above theorem holds for Lie superalgebras as well. This in particular proves the Kac-Roan-Wakimoto conjecture '03.

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- characters of all (ordinary) representations of W-algebras $\mathcal{W}^k(\mathfrak{sl}_n, f)$ of type A ([T.A.'12]), which in particular proves the similar conjecture of Kac-Wakimoto '08.

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Theorem (Zhu '96)

Let V be a "nice" vertex (operator) algebra. Then the character $\chi_M(e^{2\pi i\tau})$ converges to a holomorphic function on the upper half plane for any $M \in \mathrm{Irrep}(V)$. Moreover, the space spanned by the characters $\chi_M(e^{2\pi i\tau})$, $M \in \mathrm{Irrep}(V)$, is invariant under the natural action of $SL_2(\mathbb{Z})$.

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- V is rational, that is, any positively graded V-modules are completely reducible.

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$$\operatorname{gr} V_k(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}]) = \mathbb{C}[J_\infty \mathfrak{g}^*].$$

Here $J_\infty X$ is the arc space of X:
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$$\operatorname{gr} V_k(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}]) = \mathbb{C}[J_\infty \mathfrak{g}^*].$$

Here $J_\infty X$ is the arc space of X:
$\operatorname{Hom}(\operatorname{Spec} R, J_\infty X) = \operatorname{Hom}(\operatorname{Spec} R[[t]], X)$, $R : \mathbb{C}-$algebra.

Let $L_k(\mathfrak{g})$ be the simple (graded) quotient $L(k\Lambda_0)$ of $V^k(\mathfrak{g})$ (simple affine vertex algebra).

Fact (Frenkel-Zhu '92, Zhu '96, Dong-Mason '06)

$L_k(\mathfrak{g})$ is lisse $\iff$ $L_k(\mathfrak{g})$ is integrable ($\iff$ $k \in \mathbb{Z}_{\geq 0}$).

If this is the case,
$L_k(\mathfrak{g})$-Mod $= \{$integrable $\widehat{\mathfrak{g}}$-modules of level $k\}$. Thus, $L_k(\mathfrak{g})$ is rational as well.

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2). $X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathcal{S}_f := f + \mathfrak{g}^e \subset \mathfrak{g} = \mathfrak{g}^*$, the Slodowy slice at f ([De-Sole-Kac '06]), where $\{e, f, h\}$ is an $\mathfrak{sl}_2$-triple.

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(ii). *If* $X_{L_k(\mathfrak{g})} = \overline{G.f}$, $X_{H^0_{DS,f}(L_k(\mathfrak{g}))} = \{f\}$. *Hence* $H^0_{DS,f}(L_k(\mathfrak{g}))$ *is lisse, and so is its quotient* $\mathcal{W}_k(\mathfrak{g}, f)$.

Admissible representations of affine Kac-Moody algebras

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There is a nice class of representations of $\widehat{\mathfrak{g}}$ which are called admissible representations (Kac-Wakimoto '88):

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The simple affine vertex algebra $L_k(\mathfrak{g})$ is admissible as a $\widehat{\mathfrak{g}}$-module iff

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1, \\ h & \text{if } (q, r^\vee) = r^\vee. \end{cases}$$

Here h is the Coxeter number of $\mathfrak{g}$ and $r^\vee$ is the lacity of $\mathfrak{g}$.

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Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra, and let $f \in \mathbb{O}_k$. Then the simple affine W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse.

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An admissible affine vertex algebra $L_k(\mathfrak{g})$ is called *non-degenerate* if

$$X_{L_k(\mathfrak{g})} = \mathcal{N} = \overline{G.f_{prin}}.$$

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For $\mathfrak{g} = \mathfrak{sl}_2$, the corresponding rational W-algebras are exactly the minimal series of the Virasoro (vertex) algebra.

Adamović-Milas conjecture

The proof of the previous theorem is based on the following assertion on admissible affine vertex algebras.

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Theorem (T.A. '16, Adamović-Milas conjecture '95)

Let $L_k(\mathfrak{g})$ *be an admissible affine vertex algebra. Then* $L_k(\mathfrak{g})$ *is rational in the category* $\mathcal{O}$*, that is, any* $L_k(\mathfrak{g})$*-module that belongs to* $\mathcal{O}$ *is completely reducible.*

4d-2d duality

$4d$-$2d$ duality

Recently, Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees '15 have constructed a remarkable map

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such that, among other things, the character of the vertex algebra $\Phi(\mathcal{T})$ coincides with the Schur index of the corresponding 4d $N = 2$ SCFT $\mathcal{T}$, which is an important invariant of the theory $\mathcal{T}$.

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The main examples of vertex algebras considered by Rastelli *et al.* '15. are the simple affine vertex algebras $L_k(\mathfrak{g})$ of types D_4, E_6, E_7, E_8 at level $k = -h^\vee/6 - 1$, which are non-rational, non-admissible affine vertex algebras at negative integer levels.

Higgs branch conjecture

There is another important invariant of a 4d $N = 2$ SCFT $\mathcal{T}$, called the Higgs branch. The Higgs branch $Higgs_{\mathcal{T}}$ is an affine algebraic variety that has a hyperKähler structure in its smooth part. In particular, $Higgs_{\mathcal{T}}$ is a (possibly singular) symplectic variety.

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Let $\mathcal{T}$ be one of the 4d $N = 2$ SCFTs such that $\Phi(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = h^\vee/6 - 1$ for types D_4, E_6, E_7, E_8 appeared previously. It is known that $Higgs_{\mathcal{T}} = \overline{\mathbb{O}_{min}}$, and it turned out that this equals to the associated variety $X_{\Phi(\mathcal{T})}$ ([T.A.-Moreau '18]).

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For *any* 4d $N = 2$ SCFT $\mathcal{T}$, we have

$$Higgs_{\mathcal{T}} = X_{\Phi(\mathcal{T})}.$$

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1. Higgs branch conjecture is a physical conjecture since the Higgs branch is not mathematically defined in general. The Schur index is not a mathematically defined object in general, either.

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Remark

1. Higgs branch conjecture is a physical conjecture since the Higgs branch is not mathematically defined in general. The Schur index is not a mathematically defined object in general, either.
2. There is a close relationship between the Higgs branches of 4d $N = 2$ SCFTs and the Coulomb branches of three-dimensional $N = 4$ gauge theories whose mathematical definition has been given by Braverman-Finkelberg-Nakajima '16 (4d-3d duality).

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- The simple affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$. In particular, admissible affine vertex algebras are quasi-lisse.
- Physical intuition expects that vertex algebras that come from 4d $N = 2$ SCFTs via the map Φ are quasi-lisse.

Modulaity of Schur indices

Theorem (T.A.-Kawasetsu'16)

Let V be a quasi-lisse vertex (operator) algebra (of CFT type). Then there are only finitely many simple ordinary V-modules. Moreover, for a finitely generated ordinary V-module M, the character $\chi_M(q)$ satisfies a modular linear differential equation *(MLDE).*

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The theory of class $\mathcal{S}$

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Rastelli *et al.* '15 conjectured that chiral algebras of class $\mathcal{S}$ can be also described in terms of 2d TQFT (see [Tachikawa] for a mathematical exposition of their conjecture and the background).

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From a result of Arkhipov-Gaitsgory one finds that the identity morphism id_G is the algebra $\mathcal{D}_G^{ch}$ of *chiral differential operators* on G at the critical level, whose associated variety is canonically isomorphic to T^*G.

Higgs branch conjecture for class $\mathcal{S}$ theory

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Theorem (T.A., to appear, conjectured by Rastelli et al.)

Let $\mathbb{B}_2$ be the category of 2-bordisms. For each semisimple group G, there exists a unique monoidal functor

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which sends (1) the object S^1 to G, (2) the cylinder, which is the identity morphism id_{S^1}, to the identity morphism $\mathrm{id}_G = \mathcal{D}_G^{ch}$, and (3) the cap to $H^0_{DS,f_{prin}}(\mathcal{D}_G^{ch})$.

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$$X_{\eta_G(B)} \cong \eta_G^{BFN}(B)$$

for any 2-bordism B, where η_G^{BFN} is the functor form $\mathbb{B}_2$ to the category of symplectic varieties constructed by Braverman-Finkelberg-Nakajima '17.

Thank you!