Hitchin type moduli stacks in automorphic representation theory

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Hitchin moduli stack

- G: complex reductive group; $\mathfrak{g} = \text{Lie}(G)$.
- X: compact Riemann surface; ω_X : canonical bundle.
- A G-Higgs bundle on X is a pair $(\mathcal{E}, \varphi)$ where
  1. $\mathcal{E}$: a principal G-bundle on X;
  2. φ: a section of the vector bundle $\text{Ad}(\mathcal{E}) \otimes \omega_X$
     $\big($When $G = \text{GL}_n$, $\mathcal{E} \leftrightarrow \mathcal{V}$ v.b. of rank n, $\varphi : \mathcal{V} \to \mathcal{V} \otimes \omega_X\big)$.
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Key structures of $\mathcal{M}_G$

- (the stable part of) $\mathcal{M}_G$ admits a hyperKähler structure.
- $\mathcal{M}_G$ is essentially $T^*\mathrm{Bun}_G$ (holomorphic symplectic).
- The Hitchin map

$$f_G : \mathcal{M}_G \to \mathcal{A}_G.$$

  where $\mathcal{A}_G \cong \prod_{i=1}^{d} \Gamma(X, \omega_X^{\otimes d_i})$, if $f_1, \cdots, f_r$ are the homogeneous free generators of the ring $\mathbb{C}[\mathfrak{g}]^G$ of degree $d_1, \cdots, d_r$.
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Variants

- Group version of $\mathcal{M}_G$ (relevant to the Arthur-Selberg trace formula).
- Note that

$$\mathcal{M}_G = \{s : X \to [\mathfrak{g}/G \times \mathbb{G}_m] | \cdots\}.$$

Replace $[\mathfrak{g}/G \times \mathbb{G}_m]$ by other quotient stacks? (relevant to the relative trace formula)

- For example, replace $[\mathfrak{g}/G \times \mathbb{G}_m]$ by

$$(\mathrm{End}(V) \times V^*)/(\mathrm{GL}(V) \times \mathbb{G}_m)$$

The moduli stack of maps from X are related to the Hilbert scheme of points on planar curves. ($\rightsquigarrow$ Maulik-Y., Macdonald formula....)

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Roles in geometric representation theory

- $\mathcal{M}_G$ is essentially $T^*\mathrm{Bun}_G$.
- Geometric Langlands: Beilinson-Drinfeld, Laumon, $\cdots$.
- Char p: Bezrukavnikov-Braverman, T.H.Chen-X.Zhu.
- Global Springer theory with applications to double affine Hecke algebras (Y., Oblomkov-Y.)
- Applications to classical Langlands: Ngô's proof of the fundamental lemma (most relevant to this talk).

§2 Hitchin moduli stacks and the Arthur-Selberg trace formula

Automorphic forms

- $X/k = \mathbb{F}_q$ curve (projective, smooth, geometrically connected).
- Global function field $F = k(X)$.
- Local function field $F_x \supset \mathcal{O}_x \twoheadrightarrow k(x)$; ϖ_x a uniformizer.
- Ring of adèles $\mathbb{A} = \prod'_{x \in |X|} F_x$.
- G: split reductive group over k.
- Automorphic forms: smooth functions on $G(F)\backslash G(\mathbb{A})$.
- Cuspidal automorphic representations (when G is semisimple): irreducible $G(\mathbb{A})$-subrepresentations of $C_c^\infty(G(F)\backslash G(\mathbb{A}))$.

Hecke operators

- Fix a compact open subgroup $K \subset G(\mathbb{A})$ (level).
- The space $\mathcal{A}_K = C_c(G(F)\backslash G(\mathbb{A})/K)$ is acted upon by the Hecke algebra $\mathcal{H}_K = C_c^\infty(K\backslash G(\mathbb{A})/K)$.
- Goal: understand the $\mathcal{H}_K$-module $\mathcal{A}_K$.
- The action of $f = \mathbf{1}_{KgK} \in \mathcal{H}_K$ on $\mathcal{A}_K$ is given by the pull-push operator $q_{0!}p_0^*$ along the correspondence

$$G(F)\backslash G(\mathbb{A})/K \xleftarrow{p_0} G(F)\backslash G(\mathbb{A})/(K \cap gKg^{-1}) \xrightarrow{q_0} G(F)\backslash G(\mathbb{A})/K$$

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Arthur-Selberg trace formula

Let $f \in \mathcal{H}_K$. Arthur-Selberg trace formula expresses the trace of f on $\mathcal{A}_K$ in two different ways [1]:

- The geometric expansion

$$\mathrm{Tr}(f, \mathcal{A}_K) \text{``}=\text{''} \sum_{\gamma \in G(F)/\sim} J_\gamma(f).$$

  Here, $J_\gamma(f)$ is the orbital integral

$$J_\gamma(f) = \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A}), \mu_{G_\gamma}) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g) \frac{\mu_G}{\mu_{G_\gamma}}(g)$$

  μ_G and μ_{G_γ} are Haar measures on $G(\mathbb{A})$ and $G_\gamma(\mathbb{A})$.

- The spectral expansion: a sum over automorphic representations.

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Geometric interpretation

Goal:

$$\begin{array}{ccccc}
G(F)\backslash G(\mathbb{A})/K & \xleftarrow{p_0} & G(F)\backslash G(\mathbb{A})/K \cap gKg^{-1} & \xrightarrow{q_0} & G(F)\backslash G(\mathbb{A})/K \\
\text{Geometrize} \Big\downarrow & & \text{Geometrize} \Big\downarrow & & \text{Geometrize} \Big\downarrow \\
\mathsf{Bun}_{G,K} & \xleftarrow{\quad p \quad} & \mathsf{Hk}_{G,KgK} & \xrightarrow{\quad q \quad} & \mathsf{Bun}_{G,K}
\end{array}$$

Geometric interpretation

- Let $K_0 = \prod_{x\in|X|} G(\mathcal{O}_x)$. Weil's observation: full embedding of groupoids

$$G(F)\backslash G(\mathbb{A})/K_0 \hookrightarrow \mathrm{Bun}_G(k) = \{\text{principal } G\text{-bundles over } X\}.$$

- When G is split, this is in fact an equivalence.
- Level structures: for a compact open $K = \prod_{x\in|X|} K_x \subset K_0$, $G(F)\backslash G(\mathbb{A})/K$ is equivalent to the groupoid of G-bundles on X with K-level structures.

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Geometric interpretation

- Double coset $KgK \subset G(\mathbb{A})$ specifies the relative position of two G-bundles with K-level structures whose generic fibers are identified.
- Example: $G = \mathrm{GL}_n$, $K = K_0$, $g = (g_x)$ where $g_x = \mathrm{diag}(\varpi_x, 1, 1, \cdots, 1)$ for $x = x_0$ and $g_x = 1$ otherwise.
  $\mathcal{V}$ and $\mathcal{V}'$: two vector bundles of rank n (GL_n-bundles $\iff$ rank n vector bundles).
  A rational map $\varphi : \mathcal{V} \dashrightarrow \mathcal{V}'$ is in relative position KgK $\iff$ φ extends to an injective map of coherent sheaves $\mathcal{V} \to \mathcal{V}'$ whose cokernel is the skyscraper $k(x_0)$.

Geometric interpretation

We introduce some moduli stacks.

- $\mathrm{Bun}_{G,K}$: moduli stack of G-bundles on X with K-level structures.
- $\mathrm{Hk}_{G,KgK}$: moduli stack of triples $(\mathcal{E},\mathcal{E}',\alpha)$ where $\mathcal{E},\mathcal{E}'$ are G-bundles with K-level structures on X, and $\alpha:\mathcal{E}\dashrightarrow\mathcal{E}'$ is a *rational* isomorphism between $\mathcal{E}$ and $\mathcal{E}'$ with relative position given by KgK.
- Hecke correspondence diagram

$$\mathrm{Bun}_{G,K} \xleftarrow{p} \mathrm{Hk}_{G,KgK} \xrightarrow{q} \mathrm{Bun}_{G,K}$$

$$\mathcal{E} \longleftarrow\!\!\dashv (\mathcal{E},\mathcal{E}',\alpha) \longmapsto \mathcal{E}'$$

Taking k-points, recover the previous Hecke diagram.

$\mathcal{M}_{G,KgK}$

- Define the stack $\mathcal{M}_{G,KgK}$ by the Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_{G,KgK} & \longrightarrow & \mathsf{Hk}_{G,KgK} \\ \downarrow & & \downarrow {\scriptstyle (p,q)} \\ \mathsf{Bun}_{G,K} & \xrightarrow{\Delta} & \mathsf{Bun}_{G,K} \times \mathsf{Bun}_{G,K} \end{array}$$

- “Trace equals integrating the kernel function over the diagonal” $\Rightarrow$

$$\mathrm{Tr}(\mathbf{1}_{KgK}, \mathcal{A}_K)\text{“}=\text{”}\#\mathcal{M}_{G,KgK}(k).$$

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$\mathcal{M}_{G,KgK}$ and the trace formula

- $\mathcal{M}_{G,KgK}$ classifies $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a G-bundle over X with K-level structures, and $\varphi : \mathcal{E} \dashrightarrow \mathcal{E}$ is a rational automorphism with relative position given by KgK.
- $\mathcal{M}_{G,KgK}$ is a group version of $\mathcal{M}_G$.
- How to see the geometric expansion geometrically?
- It comes from an analogue of the Hitchin map

$$h_G : \mathcal{M}_{G,KgK} \to \mathcal{B}_{G,KgK}.$$

sending $(\mathcal{E}, \varphi)$ to invariants of φ_η; $\mathcal{B}_{G,KgK}$ an affine scheme over k.

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Hitchin map

- Grothendieck-Lefschetz trace formula implies for any $b \in \mathcal{B}_{G,KgK}(k)$

$$\sum_{\gamma \in G(F)/\sim, \chi(\gamma)=b} J_\gamma(\mathbf{1}_{KgK}) \text{``}=\text{''} \# h_G^{-1}(b) = \mathsf{Tr}(\mathsf{Frob}_b, (\mathbf{R}h_! \mathbb{Q}_\ell)_b).$$

stable orbital integrals $\leftrightarrow$ cohomology

Summing over $b \in \mathcal{B}_{G,KgK}(k)$ gives the geometric expansion of the trace.

- Such an interpretation of the orbital integrals is the starting point of B.C.Ngô's proof of the Langlands-Shelstad fundamental lemma.
- For more, see E.Frenkel, B-C.Ngo, Geometrization of trace formulas.

§3 Hitchin moduli stacks and relative trace formulae

Relative trace formula

- Let H_1, H_2 be subgroups of G.

$$\begin{array}{ccccc} H_1 & \longrightarrow & G & \longleftarrow & H_2 \\ & & \wr\downarrow & & \\ H_1(F)\backslash H_1(\mathbb{A})/K_1 & \xrightarrow{\varphi_1} & G(F)\backslash G(\mathbb{A})/K & \xleftarrow{\varphi_2} & H_2(F)\backslash H_2(\mathbb{A})/K_2 \end{array}$$

(Here $K_i = H_i(\mathbb{A}) \cap K$.)

- The relative trace of $f \in \mathcal{H}_K$ with respect to (H_1, H_2):

$$\mathrm{RTr}^G_{H_1,H_2}(f) = \langle \varphi_{1,!}\mathbf{1}, f \cdot \varphi_{2,!}\mathbf{1} \rangle_{L^2(G(F)\backslash G(\mathbb{A})/K)}$$

Here $\varphi_{i,!}\mathbf{1}$ is the pushforward of the constant function (measure) along φ_i.

Variant of Hitchin moduli

- $\mathrm{RTr}^{G}_{H_1,H_2}(f)$ also has:
  1. geometric expansion: (relative) orbital integrals;
  2. spectral expansion: periods of automorphic forms.
- Geometric interpretation

$$\mathcal{M}_{G,KgK} \rightsquigarrow \mathcal{M}^{G}_{H_1,H_2,KgK}$$

  the latter classifies $(\mathcal{E}_1, \mathcal{E}_2, \alpha)$ where
  1. $\mathcal{E}_i$: an H_i-bundle with K_i-structure over X for $i = 1, 2$;
  2. α: a rational isomorphism between the G-bundles induced from $\mathcal{E}_1$ and $\mathcal{E}_2$, with relative position given by KgK.
- Special case ($G = G_1 \times G_1, H_1 = H_2 = \Delta(G_1)$) recovers the Arthur-Selberg trace for G_1.

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The Jacquet-Rallis relative trace formula

- Motivation: Gan-Gross-Prasad conjecture for unitary groups. (Branching law)
- Jacquet and Rallis consider two situations. Let F be a global field, E/F a quadratic extension ($\eta : \mathbb{A}_F^\times \to \{\pm 1\}$ corresponds to E/F.)
  1. $G = R_{E/F}\mathrm{GL}_n \times R_{E/F}\mathrm{GL}_{n-1}$,
     $H_1 = \Delta(R_{E/F}\mathrm{GL}_{n-1})$,
     $H_2 = \mathrm{GL}_n \times \mathrm{GL}_{n-1}$ (all over F).
     The constant function on $H_2(\mathbb{A}_F)$ is replaced by the character $\eta \circ \det$ on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$).
  2. $G' = \mathrm{U}_n \times \mathrm{U}_{n-1}$,
     $H'_1 = H'_2 = \Delta(\mathrm{U}_{n-1})$.
     Here U_{n-1} is the unitary group associated with a Hermitian E-space V_{n-1} of dimension $n-1$; U_n is the one associated to $V_n = V_{n-1} \oplus E \cdot e_n$, $(e_n, e_n) = 1$.
- Compare $\mathrm{RTr}^{G}_{H_1,H_2}$ and $\mathrm{RTr}^{G'}_{H'_1,H'_2}$.

The Jacquet-Rallis orbital integrals

- The local orbital integral relevant to the first situation is

$$J^{\mathsf{GL}}_{x,\gamma}(f) := \int_{\mathsf{GL}_{n-1}(F_x)} f(h^{-1}\gamma h)\eta_x(\det h)dh, \gamma \in \mathsf{S}_n(F_x),$$

$$f \in C_c^\infty(\mathsf{S}_n(F_x)), \text{ where } \mathsf{S}_n = \{g \in R_{E/F}\mathsf{GL}_n | \sigma(g) = g^{-1}\}.$$

- The local orbital integral relevant to the second situation is

$$J^{\mathsf{U}}_{x,\delta}(f) = \int_{\mathsf{U}_{n-1}(F_x)} f(h^{-1}\delta h)dh, \quad \delta \in \mathsf{U}_n(F_x), f \in C_c^\infty(\mathsf{U}_n(F_x)).$$

The fundamental lemma

Theorem (Y., 2011)

Assume F is a function field, x is a place of F such that E_x/F_x is unramified and the Hermitian space $V_{n,x}$ has a self-dual lattices $\Lambda_{n,x}$. Then for strongly regular semisimple elements $\gamma \in \mathsf{S}_n(F_x)$ and $\delta \in \mathsf{U}_n(F_x)$ with the same invariants, we have

$$J^{\mathsf{GL}}_{x,\gamma}(\mathbf{1}_{\mathsf{S}_n(\mathcal{O}_x)}) = \pm J^{\mathsf{U}}_{x,\delta}(\mathbf{1}_{\mathsf{U}(\Lambda_{n,x})})$$

for an explicitly defined sign depending on the invariants of γ.

Proof: first reduce to the Lie algebra version, then study the geometry of Hitchin-like fibrations attached to $\mathcal{M}^G_{H_1,H_2}$ and $\mathcal{M}^{G'}_{H'_1,H'_2}$.

Application

- J.Gordon has extended the above theorem to mixed characteristic local fields, using model theory.
- W.Zhang used the J-R relative trace formulae and the fundamental lemma proved above to prove the Gan-Gross-Prasad conjecture for unitary groups (under some local conditions).

§4 Hitchin moduli stacks and Shtukas

Moduli of Shtukas

$$\frac{\text{Shimura varieties}}{\text{Number Fields}} :: \frac{\text{Moduli of Shtukas (one leg; minuscule)}}{\text{Function Fields}}$$

- Drinfeld introduced the notion of Shtukas, generalizing his elliptic modules. He used Shtukas to prove the global Langlands correspondence for GL_2 over function fields.
- Varshavsky introduced G-Shtukas for a reductive group G.
- L.Lafforgue used Drinfeld's Shtukas to prove the global Langlands correspondence for GL_n over function fields.
- V.Lafforgue used G-Shtukas to prove the "automorphic to Galois" direction of the global Langlands correspondence for reductive groups G over function fields.

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Definition of moduli of Shtukas

The definition is similar to that of $\mathcal{M}_{G,KgK}$: just replace the diagonal map by the graph of Frobenius.

$$\begin{array}{ccc} \mathrm{Sht}^{\mu}_{G,K} & \longrightarrow & \mathrm{Hk}^{\mu}_{G,K} \\ \downarrow & & \downarrow (p_0, p_r) \\ \mathrm{Bun}_{G,K} & \xrightarrow{(\mathrm{id},\mathrm{Fr})} & \mathrm{Bun}_{G,K} \times \mathrm{Bun}_{G,K} \end{array}$$

- r: the number of legs.
- $\mu = (\mu_1, \cdots, \mu_r)$, μ_i dominant coweights,
- $\mathrm{Hk}^{\mu}_{G,K}$ classifies chains of modifications at legs x_i

$$\mathcal{E}_0 \overset{f_1}{\underset{x_1}{-\,-\!\!\!\rightarrow}} \mathcal{E}_1 \overset{f_2}{\underset{x_2}{-\,-\!\!\!\rightarrow}} \cdots \overset{f_r}{\underset{x_r}{-\,-\!\!\!\rightarrow}} \mathcal{E}_r$$

where the relative position of f_i is bounded by μ_i, $1 \leq i \leq r$.

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Basic structures and properties

- Key structure: recording the legs gives a map

$$\mathrm{Sht}^{\mu}_{G,K_0} \to X^r$$

- When all $r = 0$ (so $\mu = \varnothing$), $\mathrm{Sht}^{\varnothing}_{G,K} = \mathrm{Bun}_{G,K}(k)$.
- When all μ_i are minuscule, $\mathrm{Sht}^{\mu}_{G,K}$ is smooth. (compare: Shimura varieties)
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- Key structure: recording the legs gives a map

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"Higher" automorphic forms

- Automorphic forms $\in C(\mathrm{Bun}_{G,K}(k)) = \mathrm{H}^*(\mathrm{Sht}^{\varnothing}_{G,K})$.

Definition

Higher automorphic forms are cohomology classes of $\mathrm{Sht}^{\mu}_{G,K}$ for general μ.

- Similarity: both have Hecke actions.
- Why higher? Automorphic forms appear in $\mathrm{H}^*(\mathrm{Sht}^{\mu}_{G,K})$ with interesting multiplicity spaces which carry information about the Galois side.
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Periods of automorphic representations

- Period of an automorphic representation π of $G(\mathbb{A})$ along a subgroup H is the $H(\mathbb{A})$-invariant linear functional

$$\begin{aligned} \mathcal{P}^G_{H,\pi} \quad : \quad & \pi \to \mathbb{C} \\ & \varphi \mapsto \int_{H(F)\backslash H(\mathbb{A})} \varphi \end{aligned}$$

Ansatz

Periods are related to special values of L-functions.

- Example: $G = \mathrm{GL}_2$, $H =$ diagonal torus $\rightsquigarrow$ the standard L-function of π.
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Higher periods

- Let H be a reductive subgroup of G. If the choices of λ and μ satisfy certain root-theoretic conditions, there is a natural map

$$\theta : \mathsf{Sht}_H^\lambda \to \mathsf{Sht}_G^\mu.$$

- Under further conditions, θ^* induces a map on intersection cohomology and defines a linear map

$$\mathcal{P}_{H,\lambda}^{G,\mu} : \mathsf{IH}_c^{2d_H(\lambda)}(\mathsf{Sht}_G^\mu \otimes \overline{k}) \xrightarrow{\theta^*} \mathsf{H}_c^{2d_H(\lambda)}(\mathsf{Sht}_H^\lambda \otimes \overline{k}) \xrightarrow{\int_{\mathsf{Sht}_H^\lambda}} \overline{\mathbb{Q}}_\ell.$$

Definition

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Restricting $\mathcal{P}_{H,\lambda}^{G,\mu}$ to the π-part

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is called the higher H-periods of π.

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Higher periods are related to higher derivatives of L-functions.

Let π be a (sufficiently general) cuspidal automorphic representation of $G(\mathbb{A})$. The π-part of $\mathrm{IH}_c^{2d_H(\lambda)}(\mathrm{Sht}_G^\mu \otimes \overline{k})$ is expected to be

$$\pi \otimes (\otimes_{i=1}^r \mathrm{H}^1(X \otimes \overline{k}, j_{!*}\rho_\pi^{\mu_i})).$$

1. $\pi \mapsto \rho_\pi$ is the global Langlands correspondence (ρ_π is a $\widehat{G}$-local system on Spec F);
2. $\rho_\pi^{\mu_i}$ the induced local system wrt the rep V_{μ_i} of $\widehat{G}$;
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Higher Gross-Zagier formula – setup

- $G = \mathsf{PGL}_2$, no level structure $K = K_0 = \prod_x G(\mathcal{O}_x)$.
- $r \geq 0$ an even integer.
- $\mu = (\mu_1, \cdots, \mu_r)$, where each μ_i is the minuscule coweight for G. We have the moduli stack $\mathsf{Sht}^r_G := \mathsf{Sht}^\mu_{G,K_0}$.
- X'/X: an unramified double cover.
- $\mathsf{Sht}^\lambda_T := \mathsf{Sht}^\lambda_{\mathsf{GL}_1,X'}/\mathsf{Pic}_X(k)$, the moduli of rank one Shtukas on X' with modification type $\lambda = (\lambda_1, \cdots, \lambda_r)$ ($\lambda_i = \pm 1$, $\sum_i \lambda_i = 0$), modulo twisting by line bundles from X.

Higher Gross-Zagier formula – setup

- Natural map

$$\begin{array}{ccc} \theta : \mathsf{Sht}_T^{\lambda} & \to & \mathsf{Sht}_G'^r := \mathsf{Sht}_G^r \times_{X^r} X'^r. \\ \dim : r & & 2r \end{array}$$

The image of θ gives the Heegner-Drinfeld cycle

$$Z^{\lambda} \in \mathrm{H}_c^{2r}(\mathsf{Sht}_G'^r \otimes \overline{k}, \mathbb{Q}_\ell(r)).$$

- π: everywhere unramified cuspidal automorphic representation π of $G(\mathbb{A})$,
- Z_π^{λ}: the projection of Z_π^{λ} onto the π-isotypic part $\mathrm{H}_c^{2r}(\mathsf{Sht}_G'^r \otimes \overline{k}, \mathbb{Q}_\ell(r))[\pi]$.

  making sense of this requires serious work, because the cohomology is ∞-dimensional.

Higher Gross-Zagier formula – unramified version

Theorem (Y.-Zhang, 2015)

We have

$$\langle Z^\lambda_\pi, Z^\lambda_\pi\rangle_{\mathsf{Sht}'^r_G} = \frac{q^{2-2g}}{2(\log q)^r}\frac{\mathscr{L}^{(r)}(\pi_{F'},1/2)}{L(\pi,\mathsf{Ad},1)}$$

where

- $\langle Z^\lambda_\pi, Z^\lambda_\pi\rangle_{\mathsf{Sht}'^r_G}$ *is the self-intersection number of the cycle class* Z^λ_π.
- $\pi_{F'}$ *is the base change of* π *to* $F' = k(X')$.
- $\mathscr{L}(\pi_{F'},s) = q^{4(g-1)(s-1/2)}L(\pi_{F'},s)$ *is the normalized* L*-function of* $\pi_{F'}$ *such that* $\mathscr{L}(\pi_{F'},s) = \mathscr{L}(\pi_{F'},1-s)$.
- $\mathscr{L}^{(r)}(\pi_{F'},1/2)$ *is the* r*-th derivative of* $\mathscr{L}(\pi_{F'},s)$ *at* $s=1/2$.

Remarks on the theorem

- $r = 0$: unramified case of the Waldspurger formula relating toric periods of π with central L-values.
- There is a version of the above theorem (Y.-Zhang, 2017) which allows π to have square-free levels, and X'/X is allowed to be ramified.
  In this generalization, $(-1)^r$ is the same as the sign of the functional equation for $L(\pi_{F'}, s)$.
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Comments on the proof of higher GZ

- Proof does not involve computing either side.
- Key part of the proof: comparing two relative traces

$$\begin{aligned} I(f) &= \langle Z^\lambda, f \cdot Z^\lambda \rangle_{\mathsf{Sht}'^r_G} \\ J(f,s) &= \langle \mathbf{1}_{[\mathsf{Pic}_X(k)]}, \eta|\cdot|^s_{[\mathsf{Pic}_X(k)]} \rangle_{G(F)\backslash G(\mathbb{A})/K_0}. \\ I(f) &\sim J^{(r)}(f,0). \end{aligned}$$

- Hitchin type moduli spaces play a key role in the proof. Their appearance is not due to the similarity between the moduli of Shtukas and the Hitchin stack, but "orthogonal" to this similarity.
- Similar geometric ideas can be used to prove the function field version of W.Zhang's Arithmetic Fundamental Lemma (in progress), which related derivatives of (Jacquet-Rallis) orbital integrals to intersection numbers in the moduli of unitary local Shtukas.

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