Counting vector bundles on curves

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Genus zero : Birkhoff-Grothendieck theorem

Suppose that $X = \mathbb{P}^1$. Then we have the Serre line bundles $\mathcal{O}(n), n \in \mathbb{Z}$. Moreover, $Jac(X) = \{pt\}$.

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Theorem (Birkhoff-Grothendieck)

Any indecomposable vector bundle on $\mathbb{P}^1$ is a line bundle, isomorphic to $\mathcal{O}(n)$ for some (unique) $n \in \mathbb{Z}$.

Genus one : Atiyah theorem

Suppose that X is a (smooth) elliptic curve. Then

$$Jac(X) \simeq X$$

For instance, the assignement $x \mapsto \mathcal{O}(x)$ yields an isomorphism $X \simeq Pic^1(X)$.

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Ex : For any degree one line bundle $\mathcal{O}(x)$, by Riemann-Roch we have $\mathrm{Ext}^1(\mathcal{O}(x), \mathcal{O}) = \mathbb{C}$ hence there is a unique nonsplit extension

$$0 \longrightarrow \mathcal{O}(x) \longrightarrow \mathcal{V}_x \longrightarrow \mathcal{O} \longrightarrow 0$$

Then $\{\mathcal{V}_x \mid x \in X\}$ forms a complete collection of (distinct) rank 2 and degree 1 indecomposable vector bundles on X.

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Note :

- indecomposable vector bundles are NOT semistable in general
- indecomposable vector bundles form only a constructible substack of the stack of all vector bundles

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Fact : *For any r, d, we have $A_{r,d}(X), I_{r,d}(X) < \infty$.*

For simplicity, we will only consider $A_{r,d}(X)$, which is a better behaved quantity (from which we may determine $I_{r,d}(X)$)

Some examples

The Birkhoff-Grothendieck and Atiyah theorems give respectively

$$A_{r,d}(\mathbb{P}^1) = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{if } r > 1 \end{cases}$$

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Recall that

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We may write $L_X(z) = \prod_{i=1}^{2g}(1-\sigma_i z)$, where $\{\sigma_1, \ldots, \sigma_{2g}\}$ are by definition the *Weil numbers of X*.

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In addition, we have

$$|\sigma_i| = q^{1/2}$$

for all i and we may reorder the Weil numbers so that

$$\sigma_{2i-1}\sigma_{2i} = q$$

for all i.

Weil numbers, Frobenius and $GSp(2g, \overline{\mathbb{Q}_l})$ (I)

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The étale cohomology group $H^1 = H^1(X \otimes \overline{\mathbb{F}_q}, \overline{\mathbb{Q}_l})$ carries an action of the *Frobenius element* Fr_X, and the characteristic polynomial of Fr_X^{-1} is equal to $L_X(z)$.

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H^1 also carries a symplectic (intersection) form, and Fr_X belongs to $GSp(H^1, \overline{\mathbb{Q}_l})$.

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In particular, we can evaluate any element of the character ring

$$R_g := Rep(GSp(2g, \overline{\mathbb{Q}_l}))$$

on Fr_X, for any X (defined over some field of characteristic $p \neq l$).

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All this is very explicit :

$$Rep(GSp(2g, \overline{\mathbb{Q}_l})) = \overline{\mathbb{Q}_l}[T_g]^{W_g}$$

where

$$T_g = \{(z_1, \ldots, z_{2g}) \in (\overline{\mathbb{Q}_l}^*)^{2g} \mid z_{2i-1}z_{2i} = z_{2j-1}z_{2j} \ \forall \ i, j\}$$

$$W_g = (\mathfrak{S}_2)^g \rtimes \mathfrak{S}_g.$$

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- The number of $\mathbb{F}_{q^n}$-rational points on the Jacobian of X.

$$|Jac_X(\mathbb{F}_{q^n})| = \prod_{i=1}^{2g} (1 - z_i^n),$$

etc.

Kac polynomials for curves

Theorem (S. 14')

For any $g \geq 0$ and any r, d, there exists a polynomial $A_{g,r,d} \in R_g$ such that for any smooth projective, geometrically connected curve X of genus g defined over some finite field $\mathbb{F}_q$, we have

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Moreover, Mellit proved that $A_{g,r,d}$ is independent of d (so we may just write $A_{g,r}$).

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If $r = 2$ we have

$$A_{g,2} = \prod_{i=1}^{2g} (1 - z_i) \cdot \left(\frac{\prod_i (1 - qz_i)}{(q-1)(q^2-1)} - \frac{\prod_i (1 + z_i)}{4(1+q)} + \frac{\prod_i (1 - z_i)}{2(q-1)} \left[\frac{1}{2} - \frac{1}{q-1} - \sum_i \frac{1}{1 - z_i} \right] \right).$$

A glimpse into the formula

We have

$$\sum_r \frac{A_{g,r}}{q-1} T^r = \left[(1-u)\mathrm{Log}(\Omega_g(u))\right]_{u=1}$$

where

$$\Omega_g(u) = \sum_{\mu \in \mathcal{P}} \prod_{\square \in \mu} \frac{\prod_{i=1}^{g} (u^{a(\square)+1} - z_{2i-1} q^{l(\square)})(u^{a(\square)} - z_{2i-1}^{-1} q^{l(\square)+1})}{(u^{a(\square)+1} - q^{l(\square)})(u^{a(\square)} - q^{l(\square)+1})}$$

and $a(\square)$, resp. $l(\square)$ are the arm-length and leg-length of a box in a partition.

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There are some natural generalisations of all the above to the parabolic setting (J.-A. Lin, A. Mellit). This involves Macdonald polynomials !

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For 1) $\rightsquigarrow$ Betti numbers or point count for character varieties or moduli spaces of stable Higgs bundles on curves

For 2) $\rightsquigarrow$ Hall algebras of curves and infinite-dimensional quantum groups (such as the elliptic Hall algebra), counting of cuspidal functions and function field Langlands program (for $GL(n)$)

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$$Higgs_{r,d}(X) = \{(\mathcal{E}, \theta) \mid \mathcal{E} \in Bun_{r,d}(X), \theta \in Hom(\mathcal{E}, \mathcal{E} \otimes \Omega_X)\}$$

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Let

$$\Lambda^{st}_{r,d}(X) = \{(\mathcal{E},\theta) \in Higgs^{st}_{r,d}(X) \mid \theta \text{ nilpotent}\}$$

be the stable global nilpotent cone. It is a (singular, reducible) lagrangian subvariety.

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iii) $|Irr(\Lambda^{ss}_{r,d}(X))| = A_{g,r}(0, \dots, 0)$.

Betti numbers of character varieties

For $\mathbf{k} = \mathbb{C}$, $\mathit{Higgs}^{ss}_{r,d}(X)$ is diffeomorphic to the (twisted) *character variety* $\mathcal{M}_{g,r}$ of representations of the fundamental group of a genus g Riemann surface into $GL(r)$.

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Corollary

The Poincaré polynomial of the (twisted) genus g character variety of rank r is equal to $t^{2+2(g-1)r^2} A_{g,r}(t, \ldots, t)$.

Hausel-Rodriguez-Villegas conjecture

The Poincaré polynomial of $Higgs^{ss}_{r,d}(X)$ was previously computed :
-in rank two by Hitchin ('87)
-in rank three by Gothen ('93)
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T. Hausel and F. Rodriguez-Villegas gave a conjectural formula for the Betti numbers of $Higgs^{ss}_{r,d}(X)$, later extended to $|Higgs^{ss}_{r,d}(X)(\mathbb{F}_q)|$ by S. Mozgovoy.

The main theorem together with Mellit's simplification of the explicit formula yields a proof of these conjectures.

Variations

-By a different method, one may extend the above results to moduli spaces of (semi)stable *twisted* Higgs bundles

$$Higgs^{\mathcal{L}}_{r,d}(X) = \{(\mathcal{E},\theta) \mid \mathcal{E} \in Bun_{r,d}(X),\ \theta \in Hom(\mathcal{E},\mathcal{E}\otimes\mathcal{L})\}$$

when $deg(\mathcal{L}) \leq 0$ or $deg(\mathcal{L}) \geq 2g-2$. (S. Mozgovoy, S.-, '14)

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-There is an arithmetic analogue (i.e. over number fields) by P.-H. Chaudouard (16')

Idea of proof (deformation argument)

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1. $|\mathcal{Y}_t(\mathbb{F}_q)| = |\mathcal{Y}_0(\mathbb{F}_q)|$ for any t,
2. there is a map $\pi \ : \mathcal{Y} \to Bun_{r,d}(X)$ such that, for $t \neq 0$,

$$\pi_{|\mathcal{Y}_t} \ : \mathcal{Y}_t \to Bun_{r,d}(X)$$

is a linear stack over the substack of indecomposable vector bundles.

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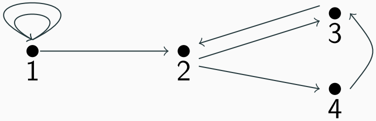

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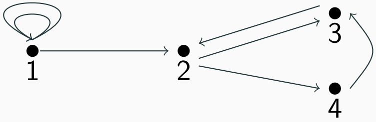

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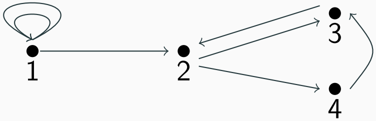

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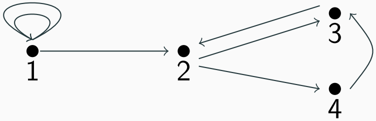

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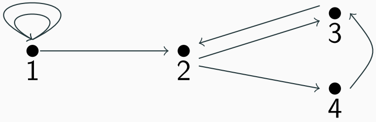

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Representations of Q over k form an abelian category.

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In both cases, these come equipped with their respective Euler forms

$$\langle M, N \rangle = hom(M, N) - ext^1(M, N)$$

Kac and Okounkov conjectures (I)

Let $Q = (I, \Omega)$ be a quiver. Let $\mathbf{d} \in \mathbb{N}^I$ be a dimension vector. For $k = \mathbb{F}_q$ a finite field, let $A_{Q,\mathbf{d},k}$ be the number of geometrically indecomposable representations of Q over k of dimension $\mathbf{d}$.

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There exists a unique polynomial $A_{Q,\mathbf{d}} \in \mathbb{Z}[t]$ such that for any k we have $A_{Q,\mathbf{d},k} = A_{Q,\mathbf{d}}(|k|)$. Moreover, $A_{Q,\mathbf{d}}(t) \in \mathbb{N}[t]$.

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When Q has no edge loops, the lattice $\mathbf{L} = (\mathbb{Z}^I, (\ ,\ ))$ is a *Kac-Moody lattice* (i.e. the root lattice of a Kac-Moody algebra). Moreover (Kac's theorem)

$$A_{Q,\mathbf{d}}(t) \neq 0 \Leftrightarrow \mathbf{d} \text{ is a root of } \mathbf{L}.$$

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Moreover, (quantum deformations of) both of these Lie algebras should be realized as appropriate Hall algebras of a 'generic' curve of genus g (a joint project with F. Sala).

Counting cuspidals

Let X be a smooth projective curve of genus g defined over $\mathbb{F}_q$. A function $f : Bun_{r,d}(X) \to \mathbb{C}$ is *cuspidal* if it is orthogonal (with respect to the orbifold pairing) to any parabolically induced function. There is a similar notion of *absolutely cuspidal* function.

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Moreover, $C_{g,r}$ is *explicitly* expressed as a polynomial in the $A_{g,s}$ for $s \leq r$.

Conjecture

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For any r, d, consider the space of simple root vectors

$$\widetilde{\mathfrak{g}}_g^{simple}[r,d] := \widetilde{\mathfrak{g}}_g[r,d] \Big/ \left(\sum_{\substack{s'+s''=r \\ d'+d''=d}} [\,\widetilde{\mathfrak{g}}_g[s',d'], \widetilde{\mathfrak{g}}_g[s'',d'']\,] \right).$$

Then $ch(\widetilde{\mathfrak{g}}_g^{simple}[r,d]) = C_{g,r}$.

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A natural question : *Is $C_{g,r}$ the cohomology of something ? (a 'cuspidal' piece of $H^*(Higgs^{ss}_{r,d}(X)$?)*

THANK YOU

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and see you at the beach