Counting vector bundles on curves

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A naive classification problem

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**Theorem (Birkhoff-Grothendieck)**

Any indecomposable vector bundle on $\mathbb{P}^1$ is a line bundle, isomorphic to $\mathcal{O}(n)$ for some (unique) $n \in \mathbb{Z}$.
Genus one: Atiyah theorem

Suppose that $X$ is a (smooth) elliptic curve. Then

$$\text{Jac}(X) \cong X$$

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**Theorem (Atiyah)**

For any $r \in \mathbb{N}$ and $d \in \mathbb{Z}$, the set of indecomposable vector bundles on $X$ of rank $r$ and degree $d$ is (canonically) isomorphic to $X$. 
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**Ex:** For any degree one line bundle $\mathcal{O}(x)$, by Riemann-Roch we have $\text{Ext}^1(\mathcal{O}(x), \mathcal{O}) = 0$ hence there is a unique nonsplit extension

$$0 \longrightarrow \mathcal{O}(x) \longrightarrow \mathcal{V}_x \longrightarrow \mathcal{O} \longrightarrow 0$$

Then $\{\mathcal{V}_x \mid x \in X\}$ forms a complete collection of (distinct) rank 2 and degree 1 indecomposable vector bundles on $X$. 
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The usual solutions:

- only consider semistable vector bundles and study the algebraic variety parametrizing them
- consider all vector bundles and study the algebraic stack parametrizing them

Note:
- indecomposable vector bundles are NOT semistable in general
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A better question

Can we count indecomposable vector bundles (of a given rank and degree)?

Let $X$ be a smooth, geometrically connected, projective curve of genus $g$ over a finite field $\mathbb{F}_q$ and let $I_{r,d}(X)$ be the number of isomorphism classes of indecomposable vector bundles on $X$ of rank $r$ and degree $d$.

$A_{r,d}(X)$ be the number of isomorphism classes of geometrically indecomposable vector bundles on $X$ of rank $r$ and degree $d$.

Fact: For any $r, d$, we have $A_{r,d}(X) < \infty$.

For simplicity, we will only consider $A_{r,d}(X)$, which is a better behaved quantity (from which we may determine $I_{r,d}(X)$).
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Some examples

The Birkhoff-Grothendieck and Atiyah theorems give respectively

\[ A_{r,d}(\mathbb{P}^1) = \begin{cases} 
1 & \text{if } r = 1 \\
0 & \text{if } r > 1 
\end{cases} \]

Recall that \( |X(F_q)| = 1 - \sigma_1 - \sigma_2 + q \) where \( \sigma_1, \sigma_2 \) are the Weil numbers of \( X \).
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Let $X$ be a smooth projective curve of genus $g$ over a finite field $\mathbb{F}_q$. 
Weil numbers (I)

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\zeta_X(z) = \exp \left( \sum_{n \geq 1} |X(\mathbb{F}_{q^n})| \frac{z^n}{n} \right).
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Theorem (Weil) There exist a (unique) polynomial $L_X(z) \in \mathbb{Z}[z]$ of degree $2g$ such that

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We may write $L_X(z) = \prod_{2g} 2 i=1 (1 - \sigma_i z)$, where \{\sigma_1, ..., \sigma_{2g}\} are by definition the Weil numbers of $X$. 


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**Weil numbers (1)**
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In addition, we have

$$|\sigma_i| = q^{1/2}$$

for all $i$ and we may reorder the Weil numbers so that

$$\sigma_{2i-1}\sigma_{2i} = q$$

for all $i$. 
Let $X$ be as before, and let $l$ be a prime number, $l \nmid q$. 
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$H^1$ also carries a symplectic (intersection) form, and $Fr_X$ belongs to $GSp(H^1, \overline{\mathbb{Q}_l})$. 
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Hence, we can view $Fr_X$ as a conjugacy class in $GSp(2g, \overline{Q_l})$.

In particular, we can evaluate any element of the character ring

$$R_g := \text{Rep}(GSp(2g, \overline{Q_l}))$$

on $Fr_X$, for any $X$ (defined over some field of characteristic $p \neq l$).
We will say that a function on the set of smooth projective curves of genus $g$ over some finite field $\mathbb{F}_q$, $l \nmid q$, is *polynomial* if it comes from a character $\chi \in R_g$. 

All this is very explicit: 

$$\text{Rep}(\text{GSp}(2g, \overline{\mathbb{Q}}_l)) = \mathbb{Q}_l[T^g]$$

where $T^g = \{(z_1, \ldots, z_{2g}) \in (\mathbb{Q}_l^*)^2 | z_{2i} - 1 z_{2i} = z_{2j} - 1 z_{2j} \forall i, j\}$

$W_g = (S_{2g}) \rtimes S_g$. 

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Weil numbers, Frobenius and $GSp(2g, \mathbb{Q}_l)$, II

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$$W_g = (S_2)^g \rtimes S_g.$$
Examples of polynomial functions

Many functions of $X$ are in fact polynomial: for instance

- The number of points of the base field $q = z_1 z_2 \cdots = z_2^g - 1 z_2^g$.
- The number of $F_q^n$-rational points on $X$ for any $n | \chi(F_q^n)$.
- The number of $F_q^n$-rational points on the Jacobian of $X$.

$|Jac_X(F_q^n)| = 2^g \prod_{i=1}^{g} (1 - z_n^i)$. etc.
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Theorem (S. 14')

For any $g \geq 0$ and any $r, d$, there exists a polynomial $A_{g,r,d} \in R_g$ such that for any smooth projective, geometrically connected curve $X$ of genus $g$ defined over some finite field $\mathbb{F}_q$, we have

$$A_{g,r,d}(X) = A_{g,r,d}(FrX).$$
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This result is effective, i.e. there is an explicit formula (later combinatorially simplified by A. Mellit).
Kac polynomials for curves

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This result is **effective**, i.e. there is an explicit formula (later combinatorially simplified by A. Mellit).

Moreover, Mellit proved that $A_{g,r,d}$ is independent of $d$ (so we may just write $A_{g,r}$).
Some examples of Kac polynomials

If \( r = 1 \) then we are simply counting points of the Jacobian \( \text{Jac}(X) \):

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If $r = 2$ we have

$$A_{g,2} = \prod_{i=1}^{2g}(1 - z_i) \cdot \left( \frac{\prod_i (1 - qz_i)}{(q - 1)(q^2 - 1)} - \frac{\prod_i (1 + z_i)}{4(1 + q)} \right)$$

$$+ \frac{\prod_i (1 - z_i)}{2(q - 1)} \left[ \frac{1}{2} - \frac{1}{q - 1} - \sum_i \frac{1}{1 - z_i} \right] \right).$$
A glimpse into the formula

We have

$$\sum_r \frac{A_{g,r}}{q-1} T^r = [(1 - u)\log(\Omega_g(u))]_{u=1}$$

where

$$\Omega_g(u) = \sum_{\mu \in \mathcal{P}} \prod_{\Box \in \mu} \frac{\prod_{i=1}^g (u^{a(\Box)} + 1 - z_{2i-1} q^{l(\Box)}))(u^{a(\Box)} - z_{2i-1}^{-1} q^{l(\Box)} + 1)}{(u^{a(\Box)} + 1 - q^{l(\Box)})(u^{a(\Box)} - q^{l(\Box)} + 1)}$$

and $a(\Box)$, resp. $l(\Box)$ are the arm-length and leg-length of a box in a partition.
**Theorem (S. 14’)**

For any $g, r,$

$$A_{g,r} \in \text{Im}(\mathbb{N}[-z_1, \ldots, -z_{2g}] \to R_g).$$
Positivity and Integrality

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**Conjecture**

\( A_{g,r} \) is integral and positive, i.e. there exists a (non virtual) \( \text{GSp}(2g, \mathbb{Q}_l) \)-representation \( A_{g,r} \) such that \( A_{g,r}(Fr_X) = \chi_{A_{g,r}}(Fr_X). \)
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There are some natural generalisations of all the above to the parabolic setting (J.-A. Lin, A. Mellit). This involves Macdonald polynomials!
Two natural questions

1) Cute, but why should I care? (i.e. do these polynomials have anything to do with anything else?)

2) What am I doing in this section? (i.e. do these polynomials have anything to do with Lie theory?)

For 1) ⇝ Betti numbers or point count for character varieties or moduli spaces of stable Higgs bundles on curves

For 2) ⇝ Hall algebras of curves and infinite-dimensional quantum groups (such as the elliptic Hall algebra), counting of cuspidal functions and function field Langlands program (for $GL(n)$)
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For 1) $\rightsquigarrow$ Betti numbers or point count for character varieties or moduli spaces of stable Higgs bundles on curves

For 2) $\rightsquigarrow$ Hall algebras of curves and infinite-dimensional quantum groups (such as the elliptic Hall algebra), counting of cuspidal functions and function field Langlands program (for $GL(n)$)
Let $X$ be a smooth projective curve of genus $g$, geometrically connected, defined over some field $k$. 

Let $\mathcal{M}_{g, r, d}(X)$ be the moduli stack of Higgs bundles on $X$ of rank $r$ and degree $d$. 

Let $\mathcal{M}_{g, r, d}^{st}(X)$ be the associated moduli space of (semi)stable Higgs bundles (a smooth quasi-projective symplectic algebraic variety). 

Let $\Lambda_{g, r, d}^{st}(X) = \{ (E, \theta) \in \mathcal{M}_{g, r, d}^{st}(X) | \theta \text{ nilpotent} \}$ be the stable global nilpotent cone. It is a (singular, reducible) lagrangian subvariety.
Let $X$ be a smooth projective curve of genus $g$, geometrically connected, defined over some field $k$. Fix coprime integers $(r, d)$. \[\text{Higgs}_{r,d}(X) = \left\{ (E, \theta) \mid E \in \text{Bun}_{r,d}(X), \theta \in \text{Hom}(E, E \otimes \Omega_X) \right\}\] be the moduli stack of Higgs bundles on $X$ of rank $r$ and degree $d$. Let $\text{Higgs}_{\text{st}}_{r,d}(X)$ be the associated moduli space of (semi)stable Higgs bundles (a smooth quasi-projective symplectic algebraic variety). Let $\Lambda_{\text{st}}_{r,d}(X) = \left\{ (E, \theta) \in \text{Higgs}_{\text{st}}_{r,d}(X) \mid \theta \text{ nilpotent} \right\}$ be the stable global nilpotent cone. It is a (singular, reducible) lagrangian subvariety.
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Let \((r, d)\) be relatively prime. The following hold:

\[|Higgs_{ss}(r, d)(\mathbb{F}_q)| = q \frac{1}{2} \left( -1 + 2 \left( \frac{g - 1}{r} \right)^2 \right) \]

\[P_{cc}(Higgs_{ss}(r, d)(X), t) = t^2 + 2 \left( \frac{g - 1}{r} \right)^2 \]

\[|\text{Irr}(\Lambda_{ss}(r, d)(X))| = \frac{1}{2} \left( -1 + 2 \left( \frac{g - 1}{r} \right)^2 \right) \]
Theorem (S., 14')

Let \((r, d)\) be relatively prime. The following hold:

i) \((k = \mathbb{F}_q)\) We have \(|\text{Higgs}_{r,d}(\mathbb{F}_q)| = q^{1+(g-1)r^2}A_{g,r}(\text{Fr}_X),\)
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2. \((k = \mathbb{C})\) The Poincaré polynomial \(P_c(\text{Higgs}^{ss}_{r,d}(X), t)\) is equal to \(t^{2+2(g-1)r^2} A_{g,r}(t, \ldots, t),\)
Betti numbers of Hitchin moduli spaces

**Theorem (S., 14’)**

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iii) \(|Irr(\Lambda^{ss}_{r,d}(X))| = A_{g,r}(0, \ldots, 0).\)
For $k = \mathbb{C}$, $Higgs_{r,d}^{ss}(X)$ is diffeomorphic to the (twisted) character variety $\mathcal{M}_{g,r}$ of representations of the fundamental group of a genus $g$ Riemann surface into $GL(r)$.
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**Corollary**

The Poincaré polynomial of the (twisted) genus $g$ character variety of rank $r$ is equal to $t^{2+2(g-1)r^2} A_{g,r}(t,\ldots, t)$. 
The Poincaré polynomial of $\text{Higgs}^{ss}_{r,d}(X)$ was previously computed:
- in rank two by Hitchin ('87)
- in rank three by Gothen ('93)
- in rank four by García-Prada, Heinloth and Schmidt ('12, algorithm)

Hausel-Rodriguez-Villegas conjecture
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T. Hausel and F. Rodriguez-Villegas gave a conjectural formula for the Betti numbers of $Higgs_{r,d}^{ss}(X)$, later extended to $|Higgs_{r,d}^{ss}(X)(\mathbb{F}_q)|$ by S. Mozgovoy.

The main theorem together with Mellit's simplification of the explicit formula yields a proof of these conjectures.
Variations

- By a different method, one may extend the above results to moduli spaces of (semi)stable twisted Higgs bundles

\[ Higgs_{r,d}^L(X) = \{(E, \theta) \mid E \in Bun_{r,d}(X), \theta \in Hom(E, E \otimes L)\} \]

when \( \deg(L) \leq 0 \) or \( \deg(L) \geq 2g - 2 \). (S. Mozgovoy, S.-, '14)

- There is a version for parabolic Higgs bundles (A. Mellit, proving a conjecture of Hausel, Letellier and Rodriguez-Villegas, '17)

- There is a motivic version, also for moduli spaces of flat connections (R. Fedorov, A. Soibelman, Y. Soibelman, 16’)

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Idea of proof (deformation argument)

Why are the numbers of indecomposable vector bundles and (semi)stable Higgs bundles of fixed rank and degree so closely related?
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One way to see this: there exists a $\mathbb{G}_m$-equivariant one-parameter family $\mathcal{Y} \to \mathbb{A}^1$

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\begin{array}{ccccccccc}
Higgs_{r,d}^{st} & \to & \mathcal{Y} & \leftarrow & \mathcal{Y}_t \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} & \to & \mathbb{A}^1 & \leftarrow & \{t\}
\end{array}
\]

with central fiber $\mathcal{Y}_0 \cong Higgs_{r,d}^{st}$ and such that:

1. $|\mathcal{Y}_t(\mathcal{F}_q)| = |\mathcal{Y}_0(\mathcal{F}_q)|$ for any $t$, 
2. there is a map $\pi: \mathcal{Y} \to \text{Bun}_{r,d}(X)$ such that, for $t \neq 0$, $\pi|\mathcal{Y}_t: \mathcal{Y}_t \to \text{Bun}_{r,d}(X)$ is a linear stack over the substack of indecomposable vector bundles.
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A *quiver* is a finite, oriented graph. Let $I$ be its set of edges and $\Omega$ its set of arrows.

Let $k$ be any field. A *representation* of $Q$ over $k$ is the following data:

i) a finite dimensional $k$-vector space $V_i$ for each $i \in I$

ii) a $k$-linear map $x_h : V_i \to V_j$ for each arrow $h : i \to j$.

Representations of $Q$ over $k$ form an abelian category.
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$$\begin{array}{c}
\bullet_{1} \xrightarrow{} \bullet_{2} \\
\downarrow \hspace{1cm}
\downarrow \\
\bullet_{3} \xleftarrow{} \bullet_{4}
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Quivers and curves

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In both cases, these come equipped with their respective Euler forms

$$\langle M, N \rangle = \text{hom}(M, N) - \text{ext}^1(M, N)$$
Let $Q = (I, \Omega)$ be a quiver. Let $d \in \mathbb{N}^I$ be a dimension vector. For $k = \mathbb{F}_q$ a finite field, let $A_{Q,d,k}$ be the number of geometrically indecomposable representations of $Q$ over $k$ of dimension $d$. 

Theorem (Kac, 81', Hausel-Letellier-Rodriguez-Villegas 13')

There exists a unique polynomial $A_{Q,d} \in \mathbb{Z}[t]$ such that for any $k$, we have $A_{Q,d,k} = A_{Q,d}(|k|)$. Moreover, $A_{Q,d}(t) \in \mathbb{N}[t]$.

When $Q$ has no edge loops, the lattice $L = (\mathbb{Z}^I, (\cdot, \cdot))$ is a Kac-Moody lattice (i.e., the root lattice of a Kac-Moody algebra). Moreover (Kac's theorem) $A_{Q,d}(t) \neq 0$ if and only if $d$ is a root of $L$. 
Kac and Okounkov conjectures (I)

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Kac conjectured (and Hausel proved later) a beautiful interpretation of $A_{Q,d}(0)$ in terms of Kac-Moody Lie algebras.

Assume for simplicity that $Q$ has no edge loops. Let $\mathfrak{g}_Q$ be the Kac-Moody algebra associated to $Q$ (i.e. with generalized Dynkin diagram equal to $Q$). It has a root space decomposition $\mathfrak{g}_Q = \mathfrak{h} \oplus \bigoplus_{d \in \Delta} \mathfrak{g}_Q^d$.

Theorem (Hausel 06', Kac conjecture 81')
For any $d \in \mathbb{N}_I$ we have $A_{Q,d}(0) = \dim(\mathfrak{g}_Q^d)$.

What about the whole polynomial $A_{Q,d}(t)$?
Maulik and Okounkov defined an $\mathbb{N}$-graded extension $\tilde{\mathfrak{g}}_Q$ of $\mathfrak{g}_Q$, using the geometry of Nakajima quiver varieties.

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**Conjecture (Okounkov)**

For any $d \in \mathbb{N}^I$ we have $A_{Q,d}(t) = \dim_{\mathbb{Z}}(\tilde{\mathfrak{g}}_Q[d]).$
Remarks (Hall algebras)

Remarks:

i) There is an extension of all of this to the setting of quivers with edge loops, for instance the quiver $S_g$ with one vertex and $g$ loops (T. Bozec).

ii) How do we construct the Kac-Moody algebra $g_Q$ from the category $\text{Rep}_FqQ$ of representations of the quiver $Q$?

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i) There is an extension of all of this to the setting of quivers with edge loops, for instance the quiver $S_g$ with one vertex and $g$ loops (T. Bozec)

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Heuristics

So, from the analogy between curves and quivers, we can expect that for any $g \geq 0$:

- There exists a natural Lie algebra $\mathfrak{g}$ with a weight decomposition $\mathfrak{g} = \bigoplus_{(r,d) \in \mathbb{Z}^2} \mathfrak{g}[r,d]$ such that for any $(r,d)$ we have $\dim(\mathfrak{g}[r,d]) = A_{\mathfrak{g},r}(0)$.

- There exists a natural Lie algebra in the category of $GSp(2g, \mathbb{Q}_l)$-modules $\tilde{\mathfrak{g}}$ with a weight decomposition $\tilde{\mathfrak{g}} = \bigoplus_{(r,d) \in \mathbb{Z}^2} \tilde{\mathfrak{g}}[r,d]$ such that for any $(r,d)$ we have $\text{ch}(\tilde{\mathfrak{g}}[r,d]) = A_{\tilde{\mathfrak{g}},r}(0)$.

Moreover, (quantum deformations of) both of these Lie algebras should be realized as appropriate Hall algebras of a 'generic' curve of genus $g$ (a joint project with F. Sala).
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Let $X$ be a smooth projective curve of genus $g$ defined over $\mathbb{F}_q$. A function $f : \text{Bun}_{r,d}(X) \to \mathbb{C}$ is \textit{cuspidal} if it is orthogonal (with respect to the orbifold pairing) to any parabolically induced function. There is a similar notion of \textit{absolutely cuspidal} function.
Counting cuspidals

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**Theorem (H. Yu 17', Deligne-Kontsevich conjecture)**

There exists a polynomial $C_{g,r} \in R_g$ such that for any $d \in \mathbb{Z}$ we have

$$\dim(Fun^{\text{abs.cusp}}(Bun_{r,d}, \mathbb{C})) = C_{g,r}(Fr_X).$$
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Moreover, $C_{g,r}$ is explicitly expressed as a polynomial in the $A_{g,s}$ for $s \leq r$. 
Conjecture

For any $r, d$, consider the space of simple root vectors

$$\tilde{g}_g^{\text{simple}}[r, d] := \tilde{g}_g[r, d]/ \left( \sum_{s'+s''=r, d'+d''=d} [\tilde{g}_g[s', d'], \tilde{g}_g[s'', d'']] \right).$$

Then $ch(\tilde{g}_g^{\text{simple}}[r, d]) = C_{g, r}$. Of course, this conjecture implies in particular that $C_{g, r}$ is integral and positive.

A natural question: Is $C_{g, r}$ the cohomology of something? (a 'cuspidal' piece of $H^\ast(Higgs_{ss} r, d(X))$?)
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THANK YOU
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and see you at the beach