On finitary approximation properties of groups and their applications

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Overview

1. Equations over groups
2. Finitary approximation properties of groups
3. Approximation by finite groups
4. Stability and examples of non-Frobenius approximable groups

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The study of equations like this goes back to:

Bernhard H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc. 18 (1943), 411.

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and a conjugate of a (namely tat^{-1}) would conjugate a to a^2. But the automorphism of $\mathbb{Z}/p\mathbb{Z}$ which sends 1 to 2 has order dividing $p-1$ and hence the order is co-prime to p.

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The resulting effect on fundamental groups is exactly

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Consider $X \subset Y \to Y/X$, with Y/X two-dimensional. When is $\pi_1(X) \to \pi_1(Y)$ injective?

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admits a covering with trivial second homology, then the system $w_1, \ldots, w_n$ *is solvable in a group containing* G.

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- When the presentation complex is aspherical. Note that in this case the number of equations k can be larger than the number of variables.
- The case when $k = n - 1$ and $\beta_1^{(2)}(Q) = 0$, i.e. the first ℓ^2-Betti number of the group Q vanishes.

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The map φ is called an (S,ε)-homomorphism.

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Sofic groups are Connes-embeddable.

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Sofic groups are Connes-embeddable.

Question (Connes, Kirchberg, etc.)

Are all groups Connes-embeddable? Are all groups MF?

Link to metric ultraproducts

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is defined as the product $\prod_{i\in I} H_i$ modulo the normal subgroup

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Proposition

A countable group is $\mathcal{C}$-approximable if and only if it embeds (as a discrete group) into a metric ultraproduct of groups from $\mathcal{C}$.

On **Sol**-approximable groups

Let **Sol** be the class of finite solvable groups.

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It follows that a finite group is **Sol**-approximable iff it is solvable.

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Finite generation is crucial here. Indeed, there exist countably infinite locally finite-p groups which are perfect and even characteristically simple. But they are **Sol**-approximable.

Approximability of topological groups

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Theorem (with Nikolov and Schneider, 2017)

The answer to Pillay's question is yes.

Main ingredient for the proofs:

Theorem (Nikolov–Segal, 2011)

Let $h_1, \ldots, h_r$ be a symmetric generating set of the finite group H. Then

$$[H,H] = \left(\prod_{i=1}^{r} [H, h_i] \right)^{e},$$

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Corollary

When H is a quotient of a product of finite groups, then for all $g, h \in H$ and $N \in \mathbb{Z}$ we have

$$[g^N, h^N] \in ([H, g][H, g^{-1}][H, h][H, h^{-1}])^e.$$

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Corollary (Arzhantseva-Paunescu, 2014)

Almost commuting permutations are close to commuting permutations.

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Proof.

Finitely generated groups which are Frobenius-approximable and Frobenius-stable are residually finite. We prove that some non-residually finite central extensions of p-adic lattices are 2-Kazhdan. Thus, they cannot be Frobenius-approximable. □

Idea of the proof of the theorem:

If $\Gamma = \langle X \mid R \rangle$ is a finitely presented group and $\varphi \colon X \to U(n)$ be a map, we set

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Thank you for your attention!