On finitary approximation properties of groups and their applications

Andreas Thom
Overview

1. Equations over groups
2. Finitary approximation properties of groups
3. Approximation by finite groups
4. Stability and examples of non-Frobenius approximable groups
How to solve a polynomial equation?

It is easy to see that \( p(t) = t^2 - 2 \) has no solution in \( \mathbb{Q} \). But there exists a solution in the field \( \mathbb{Q}[\sqrt{2}] \). In general, for any non-constant polynomial, there exists a finite field extension \( \mathbb{Q} \subset K \), such that \( p(t) = 0 \) can be solved in \( K \).

1. Consider a simple quotient \( \mathbb{Q}[t]/\langle p(t) \rangle \rightarrow K \). The image of \( t \) will satisfy the equation \( p(t) = 0 \) in \( K \).

2. Embed \( \mathbb{Q} \subset \mathbb{C} \), study the continuous map \( p: \mathbb{C} \rightarrow \mathbb{C} \), and use a topological argument to see that there exists \( \alpha \in \mathbb{C} \), such that \( p(\alpha) = 0 \).
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Equations over groups – the one variable case

Definition

Let $\Gamma$ be a group and let $g_1, \ldots, g_n \in \Gamma$, $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}$.

We say that the equation $w(t) = g_1^t \varepsilon_1 g_2^t \varepsilon_2 \ldots g_n^t \varepsilon_n$ has a solution in $\Gamma$ if there exists $h \in \Gamma$ such that $w(h) = e$.

The equation has a solution over $\Gamma$ if there is an extension $\Gamma \leq \Lambda$ and there is some $h \in \Lambda$ such that $w(h) = e$ in $\Lambda$.

The study of equations like this goes back to: Bernhard H. Neumann, Adjunction of elements to groups, J. London Math. Soc. 18 (1943), 411.
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and a conjugate of \( a \) (namely \( tat^{-1} \)) would conjugate \( a \) to \( a^2 \). But the automorphism of \( \mathbb{Z}/p\mathbb{Z} \) which sends 1 to 2 has order dividing \( p - 1 \) and hence the order is co-prime to \( p \).
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We say that the equation \( w(t) = g_1 t^{\varepsilon_1} g_2 t^{\varepsilon_2} g_3 \ldots g_n t^{\varepsilon_n} \) is non-singular if \( \sum_{i=1}^{n} \varepsilon_i \neq 0 \).
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Conjecture (Kervaire, 1960s)
If $w(t)$ is non-singular, then $w(t)$ has a solution over $\Gamma$.

Theorem (Klyachko, 1993)
If $\Gamma$ is torsionfree and $w(t)$ is non-singular, then $w(t)$ can be solved over $\Gamma$. 

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The resulting effect on fundamental groups is exactly

$$\Gamma \sim \frac{\Gamma \ast \langle t \rangle}{\langle \langle w(t) \rangle \rangle}.$$
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**Proof.**
Consider the word map $w: U(n) \to U(n)$, $w(t) = g_1 t^{\varepsilon_1} \ldots g_n t^{\varepsilon_n}$. Since $U(n)$ is connected, each $g_i$ can be moved continuously to 1. Thus, this map is homotopic to $t \mapsto t \sum \varepsilon_i$, which has non-trivial degree as a map of topological manifolds. Indeed, a generic matrix has exactly $d_n$ preimages with $d_n = |\sum \varepsilon_i|$. Hence, the map $w$ must be surjective. Each pre-image of $1$ gives a solution of the equation $w(t) = 1$. 
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Corollary (Gerstenhaber-Rothaus, 1962)

Any non-singular equation with coefficients in a finite group $\Gamma$ can be solved over $\Gamma$. 

In fact, they can be solved in a finite extension $\Gamma \leq \Lambda$. 

The same argument works for any group that is residually finite or, more generally, suitably approximated by unitary groups. Such groups are called Connes-embeddable, as they are embeddable into certain metric ultraproducts of unitary groups.

Corollary (Pestov, 2009)

Any Connes-embeddable group $\Gamma$ satisfies Kervaire’s Conjecture.

Question (Connes, 1978)

Do all groups have this approximation property?
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Equations over groups – many equations and variables

Let $w_1, w_2, \ldots, w_k \in \mathbb{F}_n \ast G$, and denote by $\varepsilon(w_1), \ldots, \varepsilon(w_k) \in \mathbb{F}_n$ their images under the natural homomorphism $\varepsilon: \mathbb{F}_n \ast G \rightarrow \mathbb{F}_n$. 
Let $w_1, w_2, \ldots, w_k \in \mathbb{F}_n * G$, and denote by $\varepsilon(w_1), \ldots, \varepsilon(w_k) \in \mathbb{F}_n$ their images under the natural homomorphism $\varepsilon: \mathbb{F}_n * G \to \mathbb{F}_n$.

**Question**

Under what conditions on $\varepsilon(w_1), \ldots, \varepsilon(w_k)$ is the homomorphism

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Question

Consider $X \subset Y \to Y/X$, with $Y/X$ two-dimensional. When is $\pi_1(X) \to \pi_1(Y)$ injective?
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It is possible to solve one equation $w \in \mathbb{F}_n \ast G$, when $\varepsilon(w) \neq 1$. 
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It is possible to solve one equation $w \in \mathbb{F}_n * G$, when $\varepsilon(w) \neq 1$.

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The conjecture holds, when $\varepsilon(w) \not\in [\mathbb{F}_n, [\mathbb{F}_n, \mathbb{F}_n]]$. 
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admits a covering with trivial second homology, then the system $w_1, \ldots, w_n$ is solvable in a group containing $G$. 
When does the theorem apply?

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- We have \( k \leq n \) and the presentation complex has itself trivial second homology. This happens when the \((n \times k)\)-matrix, whose \((i, j)\)-entry is the signed number of occurrences of the letter \( x_i \) in \( \varepsilon(w_j) \), has rank \( k \).
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- When the presentation complex is aspherical. Note that in this case the number of equations $k$ can be larger than the number of variables.

- The case when $k = n - 1$ and $\beta_1^{(2)}(Q) = 0$, i.e. the first $\ell^2$-Betti number of the group $Q$ vanishes.
Finitary approximation properties of groups

Let $C$ be a class of finite/compact metric groups, where the metric $d: G \times G \to [0,1]$ is assumed to be bi-invariant, i.e. $d(g,h) = d(fg,fh) = d(gf,hf)$ for all $f, g, h \in G$.

**Definition**

A group is said to be $C$-approximable if for any finite subset $S \subseteq G$ and $\varepsilon > 0$ there is a mapping $\varphi: S \to H$ with $H \in C$ such that

1. if $g, h, gh \in S$, then $d(\varphi(g), \varphi(h), \varphi(gh)) < \varepsilon$;
2. for $g \in S \setminus \{1\}$ we have $d(\varphi(g), 1_H) \geq 1/2$.

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Example
Groups, which are approximable
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Consider the group $U(n)$ with metrics

\[ \|u\|_{\text{HS}} = \sqrt{\frac{1}{n} \sum_{i,j=1}^{n} |u_{ij}|^2} \]

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Groups approximable w.r.t. (i)-(iii) are called Connes-embeddable, Frobenius-approximable, and MF, respectively.

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Sofic groups are Connes-embeddable.

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Link to metric ultraproducts

Definition (metric ultraproduct)

Let $\mathcal{U}$ an ultrafilter on $\mathbb{N}$ and $(H_i, d_i)_{i \in \mathbb{N}}$ a family of metric groups.

The metric ultraproduct $(H_{\mathcal{U}}, \ell)$ is defined as the product $\prod_{i \in I} H_i$ modulo the normal subgroup $N_{\mathcal{U}} := \{ (n_i) \in H | \lim_{\mathcal{U}} d_i(n_i, 1_{H_i}) = 0 \}$.

The metric $\ell$ is defined by $d([h_i], [g_i]) := \lim_{\mathcal{U}} d_i(h_i, g_i)$.

Proposition

A countable group is $C$-approximable if and only if it embeds (as a discrete group) into a metric ultraproduct of groups from $C$. 
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Every non-trivial finitely generated \textbf{Sol}-approximable has a non-trivial abelian quotient.

\textbf{Remark}
It follows that a finite group is \textbf{Sol}-approximable iff it is solvable.

Finite generation is crucial here. Indeed, there exist countably infinite locally finite-$\mathfrak{p}$ groups which are perfect and even characteristically simple. But they are \textbf{Sol}-approximable.
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Question (Doucha, 2016)
Which connected Lie groups embed topologically into a metric ultraproduct of finite groups, i.e. are \textbf{Fin}-approximable?
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Any such Lie group has abelian identity component.

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Main ingredient for the proofs:

**Theorem (Nikolov–Segal, 2011)**

Let $h_1, \ldots, h_r$ be a symmetric generating set of the finite group $H$. Then

$$[H, H] = \left( \prod_{i=1}^{r} [H, h_i] \right)^e,$$

where $e \in \mathbb{N}$ only depends on $r$. 

Corollary

When $H$ is a quotient of a product of finite groups, then for all $g, h \in H$ and $N \in \mathbb{Z}$ we have

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Let $L$ be a Lie group and $L \subset H_\mathcal{U}$ be an embedding into a metric ultraproduct. Let $a, b \in L$ be in the image of the exponential map, so there exist one-parameter subgroups $\varphi, \psi : \mathbb{R} \to L$ such that $a = \varphi(1)$ and $b = \psi(1)$. 

By continuity take $N \in \mathbb{N}$ large enough such that $g = \varphi(1/N)$ and $h = \psi(1/N)$ are $\varepsilon$-close to 1.

Now, $[a, b] = [g_N, h_N] \in ([H, g][H, g^{-1}][H, h][h^{-1}])^e$, and hence $[a, b]$ is $\varepsilon$-close to 1.

This proves the claim.
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Now, $[a, b] = [gN, hN] \in ([H, g][H, g^{-1}][H, h][h^{-1}])^e$, and hence $[a, b]$ is $8\varepsilon$-close to 1. This proves the claim.
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Definition
A group $G$ is called $C$-stable, when for all $\varepsilon > 0$, there exists $\delta > 0$, such that every $\delta$-homomorphism from $G$ to a group in $C$ is $\varepsilon$-close to a homomorphism.

Recall, an IRS on $G$ is a conjugation-invariant probability measure on the compact space of subgroups of $G$ (see the talk of Gelander).

Theorem (with Becker and Lubotzky, 2017)
An amenable group is sofic-stable if and only if all IRS are limits of finite-index IRS. This happens for example for polycyclic groups, but not for all residually finite groups.

Corollary (Arzhantseva-Paunescu, 2014)
Almost commuting permutations are close to commuting permutations.
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A group is Frobenius-stable if it is \(2\)-Kazhdan, i.e.

\[ H_2(G, H) = 0, \]

for all unitary representations \(\pi: G \to U(H)\).

Corollary

There exist groups which are not Frobenius-approximable.

Proof. Finitely generated groups which are Frobenius-approximable and Frobenius-stable are residually finite. We prove that some non-residually finite central extensions of \(p\)-adic lattices are \(2\)-Kazhdan. Thus, they cannot be Frobenius-approximable.
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Stability II

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Idea of the proof of the theorem:

If $\Gamma = \langle X \mid R \rangle$ is a finitely presented group and $\varphi : X \to U(n)$ be a map, we set

$$\text{def}(\varphi) = \max_{r \in R} \| \varphi(r) - 1_n \|.$$
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$$c(g, h) := \left[ \varphi_n(g) \varphi_n(h) - \varphi(gh) \right] \in \prod_{n \to U} M_n(C).$$

This defines a 2-cocycle on $\Gamma$ with values in a metric ultraproduct of Banach spaces. In case of the Frobenius-norm, the ultra-product is a Hilbert space and vanishing of the 2-cocycle helps to improve the sequence $(\varphi_n)_n$. More precisely, we find another sequence $(\psi_n)_n$ such that

$$\text{dist}(\varphi_n, \psi_n) = O(\text{def}(\varphi_n))$$

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$$\text{dist}(\varphi_n, \psi_n) = O(\text{def}(\varphi_n)) \quad \text{and} \quad \text{def}(\psi_n) = o(\text{def}(\varphi_n)).$$
Thank you for your attention!