Little disks operads and Feynman diagrams

Thomas Willwacher

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Goal

Topology: We want to understand

- Configuration spaces
- Embedding spaces $\mathrm{Emb}(M, N)$ of manifolds
- Diffeomorphism groups $\mathrm{Diff}(M)$

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Physics:

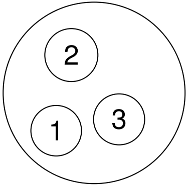
- Perturbative TQFTs (AKSZ type)

Little n-disks operad

- Space of rectilinear embeddings of n-dimensional disks

$$\mathrm{D}_n(k) = \mathrm{Emb}_{rl}(\underbrace{\mathbb{D}^n \sqcup \cdots \sqcup \mathbb{D}^n}_{k\times}, \mathbb{D}^n)$$

with $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$.



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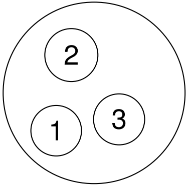
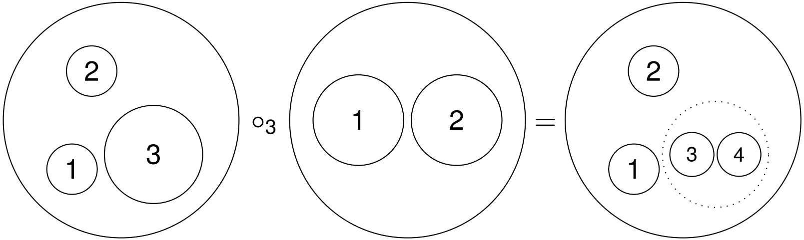


- Rectilinear: Can scale + translate, but no rotation/deformation

Little *n*-disks operad

- Can glue configuration into one another



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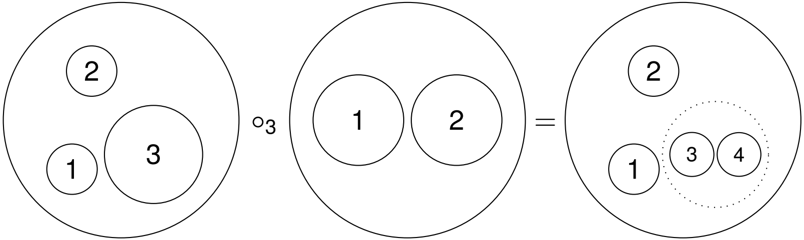

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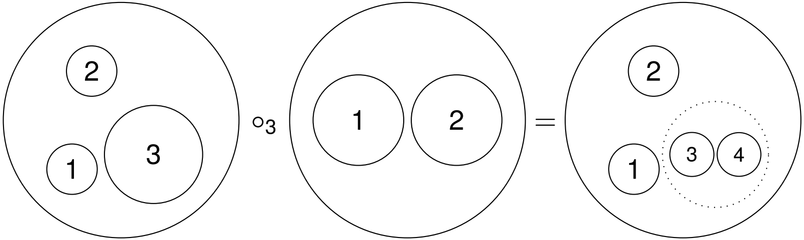

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  - $\Rightarrow$ Operad structure

Little n-disks operad

- More concretely, an operad is:
  - A collection of spaces $\mathcal{P}(r)$, with an S_r action
  - Composition maps $\circ_j : \mathcal{P}(r) \times \mathcal{P}(s) \to \mathcal{P}(r+s-1)$.
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- Very important and long studied in topology

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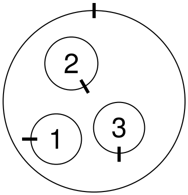
- Can allow rotation of disks → *framed* little n-disks operad D_n^{fr}.



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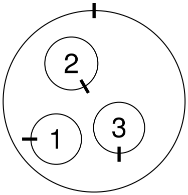

- Operadic composition (gluing) appropriately defined

Configuration spaces as modules

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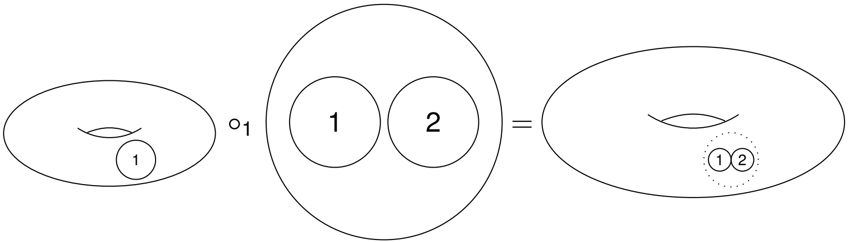
- Forms a right $\mathsf{D}_m^{\mathrm{fr}}$-module



Goodwillie-Weiss calculus

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$\Rightarrow$ Exploit these maps to understand the left-hand sides

Goodwillie-Weiss calculus

In good cases, this even yields a complete answer:

Theorem (T. Goodwillie, J. Klein, M. Weiss, P. Boavida de Brito)

For M, N manifolds, $dim(N) - dim(M) \geq 3$

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But:

$\mathrm{Map}^h_{\mathsf{D}^{\mathrm{fr}}_m - mod}(\mathrm{conf}(M), \mathrm{conf}(N))$, $\mathrm{Aut}^h_{\mathsf{D}^{\mathrm{fr}}_m - mod}(\mathrm{conf}(M))$ difficult to compute

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$$\begin{aligned} \mathrm{Emb}(M,N) &\to \mathrm{Map}^h_{\mathsf{D}^{\mathrm{fr}}_m-mod}(\mathrm{conf}(M),\mathrm{conf}(N)) \\ &\to \mathrm{Map}^h_{comod-\Omega(\mathsf{D}^{fr}_m)}(\Omega(\mathrm{conf}(N)),\Omega(\mathrm{conf}(M))) \\ \mathrm{Diff}(M) &\to \mathrm{Aut}^h_{\mathsf{D}^{\mathrm{fr}}_m-mod}(\mathrm{conf}(M)) \\ &\to \mathrm{Aut}^h_{comod-\Omega(\mathsf{D}^{fr}_m)}(\Omega(\mathrm{conf}(M))) \end{aligned}$$

Where $\Omega(\dots)$ denotes differential (or PL) forms

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- Right-hand side is completely algebraic (vector spaces)

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- Today: Discuss a case where everything works

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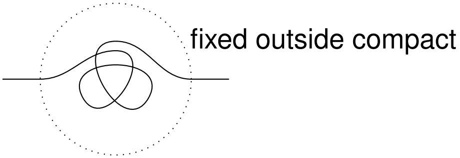
- Spaces above have models built from Feynman diagrams of TQFTs, reflecting all algebraic structures
- $\Rightarrow$ can express (in good cases) rational homotopy type combinatorially through diagrams (graph complexes)
- Similar to invariants from “actual” TQFT, but with control of the strength of the invariants
- $\Rightarrow$ Obtain non-trivial topological statements (Today: examples)

Example

Example: Rational model for spaces of higher dimensional long knots (joint w/ B. Fresse and V. Turchin)

Goodwillie-Weiss embedding calculus

- Space of embeddings (long knots) $\mathrm{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$



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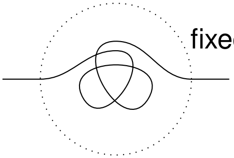

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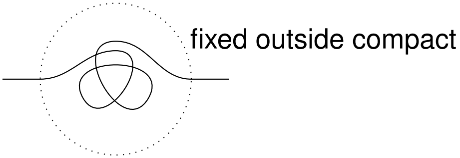

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- Classical problem: Understand homotopy groups $\pi_k(\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n))$.

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If $n - m \geq 3$ then

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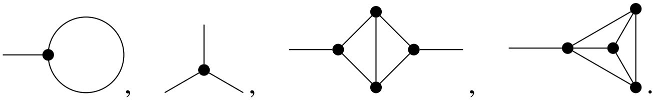
- Today: Understand the algebraic side rationally, i.e., $\pi_k(\mathrm{Map}_{\mathrm{op}}(\mathsf{D}_m, \mathsf{D}_n)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Distraction: Hairy graph complexes $\mathrm{HGC}_{m,n}$

$$\mathrm{HGC}_{m,n} = span^{gr}_{\mathbb{Q}} \{\text{isomorphism classes of admissible hairy graphs}\}$$



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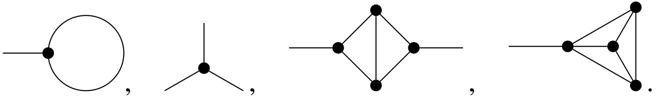

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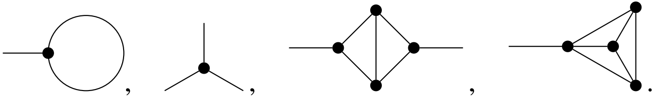

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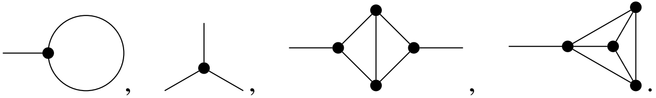

- Vertices have degree $-n$, hairs $-m$, edges $n-1$.
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- Carries natural dg Lie or L_∞-algebra structure.

Hairy graph complexes $\mathrm{HGC}_{m,n}$

- dg Lie/L_∞ structure
  - Differential δ is vertex splitting.

$$\delta\Gamma = \sum_{v \text{ vertex}} \pm \Gamma \text{ split } v$$

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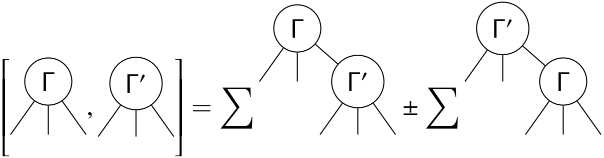
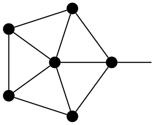
  - Lie bracket for $m > 1$



(for $m = 1$ more complicated)

Example for δ

Example for $n = 2$:



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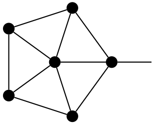

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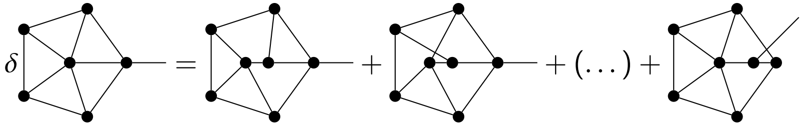

Interlude: dg Lie algebras and Maurer-Cartan spaces

- Maurer-Cartan elements of dg Lie algebra $\mathfrak{g}$

$$\mathrm{MC}(\mathfrak{g}) = \left\{ x \in \mathfrak{g}_{-1} \mid dx + \frac{1}{2}[x, x] = 0 \right\}$$

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- Maurer-Cartan space (simplicial set)

$$\mathrm{MC}_\bullet(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g} \hat{\otimes} \Omega_{poly}(\Delta^\bullet))$$

with $\Omega_{poly}(\Delta^k)$ the polynomial differential forms on a k-simplex.

Interlude: Two important results about MC spaces

Theorem (Berglund)

$\mathfrak{g}$: pro-nilpotent dg Lie (or L_∞-)algebra, $x \in \mathrm{MC}(\mathfrak{g})$. Then for $k \geq 1$

$$\pi_k(\mathrm{MC}_\bullet(\mathfrak{g}), x) \cong H_{k-1}(\mathfrak{g}^x),$$

where $\mathfrak{g}^x := (\mathfrak{g}, d + [x, -], [-, -])$ is the twisted dg Lie algebra. For $k = 1$ the rhs. is equipped with the PBW group structure.

Recall: Goodwillie-Weiss embedding calculus

Theorem (Goodwillie, Weiss, Boavida-de-Brito, Dwyer–Hess)

If $n - m \geq 3$ then

$$\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1}\mathrm{Map}^h_{op}(\mathsf{D}_m, \mathsf{D}_n).$$

Long knots

Theorem (Fresse, Turchin, W.)

For $2 \leq n \geq m \geq 1$ we have

$$\mathrm{Map}^h_{op}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}) \simeq \mathsf{MC}_\bullet(\mathsf{HGC}_{m,n}).$$

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- $\mathsf{D}_n^{\mathbb{Q}}$: rationalization of D_n, relation to previous slide:

Lemma

For $n - m \geq 3$ and $k \geq 2$:

$$\pi_k(\mathrm{Map}^h(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}})) = \pi_k(\mathrm{Map}^h(\mathsf{D}_m, \mathsf{D}_n)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consequences/Remarks

- By Goodwillie-Weiss calculus and our result:

$$\pi_k(\overline{\mathrm{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_{k+m+1}(\mathsf{MC}_\bullet(\mathsf{HGC}_{m,n})) \cong H_{k+m}(\mathsf{HGC}_{m,n})$$

  for $n - m \geq 3$. Shows that diagrams give complete rational knot invariants. (Improves earlier result by Arone-Turchin.)

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  for $n - m \geq 3$. Shows that diagrams give complete rational knot invariants. (Improves earlier result by Arone-Turchin.)
- Expresses rational homotopy type of mapping space and hence knot spaces (codim ≥ 3) through combinatorial data.

Consequences/Remarks

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  for $n - m \geq 3$. Shows that diagrams give complete rational knot invariants. (Improves earlier result by Arone-Turchin.)
- Expresses rational homotopy type of mapping space and hence knot spaces (codim ≥ 3) through combinatorial data.
- Mind: We don't have complete knowledge of $H(\mathsf{HGC}_{m,n})$.

More concrete consequences (π_0)

Theorem (essentially Haefliger '62, '66, adapted to $\overline{Emb}(\dots)$)

The $(4k-1)$-sphere knots in $6k$-space. Furthermore, if $n-m\geq 3$:

$$\pi_0(\overline{Emb}_\partial(\mathbb{R}^m,\mathbb{R}^n))\otimes\mathbb{Q}\cong\begin{cases}\mathbb{Q} & m=2k+1,\ n=4k+3 \text{ for some } k\\ \mathbb{Q} & m=4k-1,\ n=6k \text{ for some } k\\ 0 & \text{otherwise}\end{cases}.$$

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Proof.

Elementary degree counting exercise: Contributing classes in $H(\mathrm{HGC}_{m,n})$ must have loop order 0. Those are easily listed:

 and 

More concrete consequences (π_k)

- Similar degree counting + explicit knowledge of $H(\mathrm{HGC}_{m,n})$ for loop orders ≤ 2:
  $\Rightarrow$ know $\pi_k(\overline{Emb_\partial}(\mathbb{R}^m, \mathbb{R}^n)) \otimes \mathbb{Q}$ for $k \leq 3n - m - 9$.

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- Know large (infinite) families of non-trivial classes in $H(\mathrm{HGC}_{m,n})$ with interesting structure.

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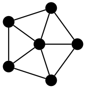
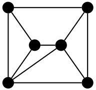
- ... well ....

Non-hairy graphs

- To understand $\mathrm{HGC}_{m,n}$, one first needs to consider simpler ”non-hairy” variant GC_n

Kontsevich's graph complexes

- differential graded vector spaces of formal $\mathbb{Q}$-linear series of graphs

 $+\frac{5}{2}$ 

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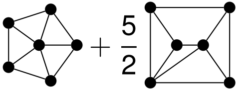

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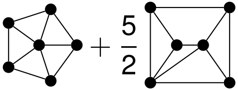

- Differential δ: vertex splitting

$$\delta\Gamma = \sum_{v \text{ vertex}} \pm \Gamma \text{ split } v$$

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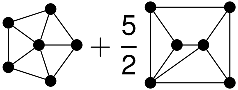

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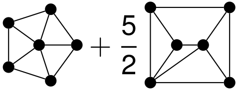

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- $\delta^2 = 0$, $\Rightarrow$ can compute graph homology $ker\delta / im\delta$.

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For $n \in \mathbb{Z}$ define

$$\mathrm{GC}_n = span^{gr}_{\mathbb{Q}} \{\text{isomorphism classes of admissible graphs}\}$$

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- $\Rightarrow$ Nontrivial stabilization

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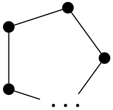
- Have classes in GC_n for $k \equiv 2n+1 \mod 4$

$W_k =$  (k vertices and k edges, deg=$n-k$)

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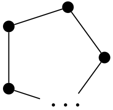
$W_k =$ 

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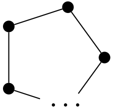
$W_k =$ 

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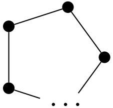
$W_k =$ 

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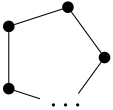
$W_k =$ 

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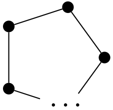
$W_k =$ 

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  - Highest degree classes have many vertices (v), few edges (e)
  - Trivalence condition: $e \geq \frac{3}{2}v$
  - $\Rightarrow$ lower bound on degree in $H(GC_n^{\geq 3-\text{valent}})$: $(\text{degree}) \geq (\#\text{loops})(n-3)+3$

Algebraic structures - plain (non-hairy) graphs

GC_n carries a dg Lie algebra structure:

$$[\gamma, \nu] = \gamma \bullet \nu - (-1)^{|\gamma||\nu|} \nu \bullet \gamma$$

with

$$\gamma \bullet \nu := \sum_{x \in V\gamma} \gamma(\text{insert } \nu \text{ in place of } x)$$

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Note:

$$\delta(-) = [\bullet\!\!-\!\!\!-\!\!\bullet, -]$$

Not so cheap results (n=2)

Theorem (T.W. '14)

$$H_0(\mathrm{GC}_2) \cong \mathfrak{grt}_1$$
$$H_1(\mathrm{GC}_2) \cong \mathbb{K}$$
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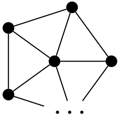
Theorem (F. Brown '12)

$$\mathit{FreeLie}(\sigma_3, \sigma_5, \sigma_7, \dots) \hookrightarrow \mathfrak{grt}_1$$

Deligne-Drinfeld conjecture: It is an isomorphism

In fact σ_{2j+1} lives in loop order $2j+1$,

$\sigma_{2j+1} =$  $+ (\cdots)$ (2j + 2 vertices and 4j + 2 edges).

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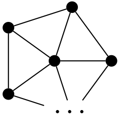
$\sigma_{2j+1} =$  $+ (\cdots)$ $(2j+2$ vertices and $4j+2$ edges).

... and there is an explicit formula for the terms $(\cdots)$ [Rossi-W.]

Computer results

$n = 2$, degree ($\downarrow$), loop order ($\rightarrow$), values $dim\, H_j(\mathrm{GC}_2)_{k\text{ loops}}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
-7	1	0	0	0	0	0	0	0	0	1				
-6	0	0	0	0	0	0	0	0	0	0				
-5	0	0	0	0	0	0	0	0	0	0				
-4	0	0	0	0	0	0	0	0	0	0				
-3	1	0	0	0	0	1	0	1	1	2				
-2	0	0	0	0	0	0	0	0	0	0				
-1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
1	1	0	0	0	0	0	0	0	0	0	0	0	0	

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- Have many nontrivial classes in $H_3(\mathrm{GC}_3)$ from Chern-Simons theory.

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- Have many nontrivial classes in $H_3(\mathrm{GC}_3)$ from Chern-Simons theory.
- (Vogel, Kneissler) Have a map (conjecturally iso)

$$\mathbb{K}[t, \omega_0, ...\omega_p, ...]/\langle \omega_p\omega_q - \omega_0\omega_{p+q}, P\rangle \to H_3(\mathrm{GC}_3)$$

Computer results

$n = 3$, degree ($\uparrow$), loop order ($\rightarrow$)

	1	2	3	4	5	6	7	8	9	10	11	12
-8	1											
-4	1											
0	1											
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	1	2	2	3	4	5	6	8	9
4	0	0	0	0	0	0	0	0				
5	0	0	0	0	0	0	0	0				
6	0	0	0	0	0	1	1	2				
7	0	0	0	0	0	0	0	0				
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- Idea: Deform differential on graph complex.

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Theorem (A. Khoroshkin, M. Živković, T.W., 2014)

There is a spectral sequence E such that $E^1 \cong H(\mathrm{GC}_n)$ and

$$E \Rightarrow \begin{cases} \mathbb{Q}[1-n] & n \text{ even} \\ \mathbb{Q}[-n] & n \text{ odd} \end{cases}$$

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"Graph homology classes come in pairs."

Other degrees

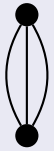
More precisely:

Theorem (A. Khoroshkin, M. Živković, T.W.)

$$H(\mathrm{GC}_0^2, \delta + [$$  $$, \cdot]) = \mathbb{Q}[1]$$

$$H(\mathrm{GC}_1^2, \delta + [m, -]) \cong \mathbb{Q}[-1]$$

where

$$m = \sum_{j \geq 1} \frac{1}{(2j+1)!}$$  $$\,{}^{2j+1}$$

Cancellations in spectral sequence (even case)

$n = 2$, degree (↑), loop order (→)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
-7	1								0	1				
-6	0							0	0	0				
-5	0						0	0	0	0				
-4	0					0	0	0	0	0				
-3	1	0	0	0	0	1	0	1	1	2				
-2	0	0	0	0	0	0	0	0	0	0				
-1	0	0	0	0	0	0	0	0	0	0	0			
0	0	0	1	0	1	0	1	1	1	1	2	2	3	
1	1	0	0	0	0	0	0	0	0	0	0	0	0	

Cancellations in spectral sequence (odd case)

$n = 3$, degree (↑), genus (→)

	1	2	3	4	5	6	7	8	9	10	11	12
-8	1											
-4	1											
0	1											
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	1	2	2	3	4	5	6	8	9
4	0	0	0	0	0	0	0	0				
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6	0	0	0	0	0	1	1	2				
7	0	0	0	0	0	0	0	0				
8	0	0	0	0	0	0	0	0	0			

In summary

- Have known series of classes in one degree + their ”partners”
- Explains all classes in $H(\mathrm{GC}_n)$ in computer accessible regime
- But: Computer cannot see very far

Hairy graphs vs non-hairy

- Have a map by adding one hair:

$$\mathrm{GC}_n^{\geq 3} \to \mathrm{HGC}_{m,n}[-m+2n-1]$$

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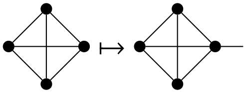

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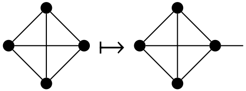

Theorem (V. Turchin, T.W.)

The above map is an injection in homology for all m, n.

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Theorem (A. K., V. T., M.Ž.,T. W.)

We have two spectral sequences E, F, such that $E^1 = H(\mathrm{HGC}_{m,n}) = F^1$ and

$$E \Rightarrow 0$$
$$F \Rightarrow H(\mathrm{GC}_n)$$

Computer data, $n = m = 2$

$\dim H(\mathrm{HGC}_{2,2})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree -3.

	1	2	3	4	5	6	7	8
9	1_{16}	1_{16}						
8		1_{15}						
7	1_{12}	1_{12}						
6		1_{11}						
5	1_8			1_{11}				
4				1_8				
3	1_4			1_7				
2								
1			1_1		1_1	1_4	1_1	$1_1, 1_4$

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2								
1			1_1		1_1	1_4	1_1	$1_1, 1_4$

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	1	2	3	4	5	6	7	8
9		1_6						
8								
7	1_5	1_5	2_5					
6			1_2					
5		1_2	2_2	3_2	5_2			
4			1_1	1_1	1_{-1}			
3	1_1	1_1	2_1	2_1	3_1			
2					1_{-2}	1_{-2}		
1			1_{-2}	1_{-2}	2_{-2}	$2_{-2}, 1_{-5}$	1_{-5}	

Computer data, $n = 3$, $m = 2$

$\dim H(\mathrm{HGC}_{2,3})$, number of hairs (↑), genus (→)
Entry 1_3 means one class in degree -3.

	1	2	3	4	5	6	7	8
9		1_6						
8								
7	1_5	1_5	2_5					
6			1_2					
5		1_2	2_2	3_2	5_2			
4			1_1	1_1	1_{-1}			
3	1_1	1_1	2_1	2_1	3_1			
2					1_{-2}	1_{-2}		
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Final remark: Role of GC_n

Theorem (B. Fresse, V. Turchin, T.W.)

GC_n *acts on a model for* $\mathsf{D}_n^{\mathbb{Q}}$ *and*

$$H(BiDer^h(\mathsf{D}_n^{\mathbb{Q}})) \cong H(\mathrm{GC}_n) \oplus \mathbb{Q}$$

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- Contains obstructions for constructing operad maps

Corollary: Intrinsic formality

Consequence of vanishing Lemmas:

Theorem (Fresse, W., Rational intrinsic formality and rigidity)

$\mathcal{C}$: any dg Hopf cooperad with $H(\mathcal{C}) \cong \mathsf{e}_n^* =: H(E_n, \mathbb{Q})$, and $n \geq 3$. Then there is a zigzag

$$\mathcal{C} \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \mathsf{e}_n^*,$$

inducing $H(\mathcal{C}) \cong \mathsf{e}_n^*$ if either (i) $4 \nmid n$ or (ii) there is an involution $J : \mathcal{C} \to \mathcal{C}$ inducing on homology the involution sending the bracket to minus itself.
This morphism is homotopically unique if $4 \nmid n - 3$, and J-equivariantly homotopically unique for all $n \geq 3$ in the presence of an involution J as above.

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Graph complex appear in several (many) important problems in homotopy theory.

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For $g \geq 2$ there is a surjection

$$H^{4g-6-k}(\mathcal{M}_g, \mathbb{Q}) \twoheadrightarrow (H^k(\mathrm{GC}_2^{(g)}))^*,$$

with $\mathrm{GC}_2^{(g)} \subset \mathrm{GC}_2$ the subcomplex of loop order g.

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- $\Rightarrow$ Nonvanishing graph cohomology yields nonvanishing cohomology classes in the moduli space of curves.

Summary

- Problems in several areas of mathematics reducible to (Feynman) graph complexes.
- Computation of graph homology hard and long standing problem.
- But: Partial results and many classes known due to recent work.
- Thanks for listening!

Peek into high loop orders

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  - For $HGC_{m,n}$ done by Arone,Turchin, Songafouo Tsopméné.

Theorem (T.W., M. Živković, Adv. in Math. '15)

Define generating functions for numbers of graphs:

$$P^{odd}(s,t) := \sum_{v,e} \dim\left(\mathrm{GC}^{odd}_{v,e}\right) s^v t^e \qquad P^{even}(s,t) := \sum_{v,e} \dim\left(\mathrm{GC}^{even}_{v,e}\right) s^v t^e .$$

There exists an explicit formula.

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There exists an explicit formula.

$$P^{odd}(s,t) := \frac{1}{\left(-s,(st)^2\right)_\infty \left((st)^2,(st)^2\right)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_\alpha \frac{(-s)^{\alpha j_\alpha}}{j_\alpha!(-\alpha)^{j_\alpha}} \frac{1}{\left((-st)^\alpha,(-st)^\alpha\right)_\infty^{j_\alpha}} \left(\frac{\left(t^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{\left((-s)^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2}\right)_\infty}\right)^{j_{2\alpha-1}/2}$$
$$\left(\frac{\left(t^\alpha,(st)^{2\alpha}\right)_\infty}{\left((-s)^\alpha t^{2\alpha},(st)^{2\alpha}\right)_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \frac{1}{\left(t^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta)j_\alpha j_\beta/2}},$$

$$P^{even}(s,t) := \frac{\left(s,(st)^2\right)_\infty}{\left(-st,(st)^2\right)_\infty} \sum_{j_1,j_2,\dots\geq 0} \prod_\alpha \frac{s^{\alpha j_\alpha}}{j_\alpha!\alpha^{j_\alpha}} \frac{1}{\left((-st)^\alpha,(-st)^\alpha\right)_\infty^{j_\alpha}} \left(\frac{\left((-t)^{2\alpha-1},(st)^{4\alpha-2}\right)_\infty}{\left(s^{2\alpha-1}t^{4\alpha-2},(st)^{4\alpha-2}\right)_\infty}\right)^{j_{2\alpha-1}/2}$$
$$\left(\frac{\left((-t)^\alpha,(st)^{2\alpha}\right)_\infty}{\left(s^\alpha t^{2\alpha},(st)^{2\alpha}\right)_\infty}\right)^{j_{2\alpha}} \prod_{\alpha,\beta} \left((-t)^{\mathrm{lcm}(\alpha,\beta)},(-st)^{\mathrm{lcm}(\alpha,\beta)}\right)_\infty^{\gcd(\alpha,\beta)j_\alpha j_\beta/2}$$

where $(a,q)_\infty = \prod_{k\geq 0}\left(1-aq^k\right)$ is the q-Pochhammer symbol.

	Even	Odd
loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$
1	0	1
2	1	1
3	0	1
4	1	2
5	-1	1
6	1	2
7	0	2
8	0	2
9	-2	1
10	1	3
11	0	1
12	0	3
13	-2	4
14	0	2
15	-4	2

	Even	Odd
loop order	$\tilde{\chi}_b^{even}$	$\tilde{\chi}_b^{odd}$
16	-3	6
17	-1	4
18	8	-5
19	12	-14
20	27	-21
21	14	-11
22	-25	21
23	-39	44
24	-496	504
25	-2979	2969
26	-412	413
27	38725	-38717
28	10583	-10578
29	-667610	667596
30	28305	-28290

The end

Thanks for listening!