Flexibility in symplectic and contact geometry

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What is symplectic geometry?

- Symplectic geometry is a geometry related to Hamiltonian dynamics, geometric topology, algebraic geometry, and string theory.
- By definition, a symplectic structure is a 2-form $\omega$ which is closed and non-degenerate. (Think: the imaginary part of a Hermitian metric.)
- An example of particular interest is holomorphic submanifolds $X \subseteq \mathbb{C}^N$ (Stein manifolds), with $\omega = i2\sum dz \wedge d\bar{z}$.
- Symplectic structures have no moduli (exact case), and no local invariants.
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A contact structure on an odd dimensional manifold $Y$ is a hyperplane field $\xi \subseteq TY$ which is maximally non-integrable.

If $X \subseteq \mathbb{C}^N$ is transverse to $S^{2N-1}$, then $Y = X \cap S^{2N-1}$ has a contact structure defined by $\xi = TY \cap iTY$.

Like symplectic structures, contact structure has no moduli and no local invariants.

$\xi$ on $Y$ determines and is determined by the structure $\omega$ on $X$, at infinity.
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Example contact structure

The standard contact structure $\xi = \ker(dz - \sum y_j dx_j)$
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Every contact manifold can be built by gluing copies of this model together via contactomorphisms.
Every question about the existence or uniqueness of geometric objects has a corresponding topological version which is weaker. “Put on algebraic topology glasses.” We use the word “formal” for these weaker questions.

For a simple example, a formal immersion $M \rightarrow N$ consists of a bundle monomorphism $TM \rightarrow TN$. 
Major question: geometry vs. topology

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Major question: geometry vs. topology

A formal symplectic structure (exact) on a smooth manifold $X$ is a complex structure on the bundle $TX$. A formal contact structure on $Y$ is a complex structure on $TY \oplus \mathbb{R}$. These structures (up to homotopy) are equivalent to lifts of $BU(n) \downarrow X, Y \uparrow \rightarrow BO(2n)$. Symplectic and contact geometry one place where this distinction is especially interesting: often the geometry gets very close to the topology!
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Existence of contact structures

Which smooth manifolds $Y$ admit contact structures? If $\eta$ is a formal contact structure on $Y$, is $\eta$ realized (up to homotopy) by a contact structure?

Theorem (Lutz 1971, Martinet 1971):
Every formal contact structure on a $3$–manifold $Y$ is realized by a contact structure.

Every formal contact structure on a $5$–manifold $Y$ is realized by a contact structure.
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Despite the publishing dates, Casals–Pancholi–Presas and Etnyre predate the above result by about two years. A feature of the proof in [BEM] is that it is not inductive in dimension.
Proof sketch

Triangulate $Y$. Building a contact structure on the codimension 1 skeleton is easy. Thus the problem reduces to extending a contact structure from a neighborhood of $\partial B^{2n-1}$ to $B^{2n-1}$.

Let $\phi^t : (D^{2n-3},\xi_{std}) \to (D^{2n-3},\xi_{std})$ be an isotopy through contactomorphisms. Because contact geometry is tied to Hamiltonian dynamics, every contact isotopy comes from an energy function, or Hamiltonian. That is, the Lie algebra of the contactomorphism group $\text{Cont}(D^{2n-3})$ is equal to $\text{cont}(D^{2n-3}) \cong C^\infty(D^{2n-3})$.

Following from the elementary theory, by suspending any contact isotopy of $D^{2n-3}$ we obtain a germ of a contact structure near $S^{2n-2}$. If the Hamiltonian generating this isotopy is positive everywhere, then the contact structure extends in an obvious way to $B^{2n-1}$.
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Proof sketch, continued

By taking our original triangulation on $Y$ to be sufficiently fine, we only need to consider contact structures on $\partial B^{2n-1}$ which are of this suspension type, thus we need to understand how the Lie group $\text{Cont}(D_{\text{std}}^{2n-3})$ interacts with the positive cone on $	ext{cont}(D_{\text{std}}^{2n-3}) \cong C^\infty(D^{2n-3})$. 

Besides the adjoint action, we have an additional "monoid structure" coming from contactomorphisms $D_{\text{std}}^{2n-3} \# D_{\text{std}}^{2n-3} \cong D_{\text{std}}^{2n-3}$ (boundary connect sum). We can realize this monoid structure by doing ambient connect sums inside $Y$, in this way it becomes an additional tool to construct extensions.

To complete the proof that any contact structure on $\partial B^{2n-1}$ extends to the interior, we prove the non-existence of any "causality" on $\text{Cont}(D_{\text{std}}^{2n-3})$, which is compatible with the positive cone in $\text{cont}(D_{\text{std}}^{2n-3})$, together with the adjoint and connect sum actions.
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Uniqueness?

Contact structures on compact manifolds have no moduli: any deformation of a contact structure is realized by an ambient isotopy. Thus, uniqueness is just parametric existence. That is, if $\xi_0$ and $\xi_1$ are contact structures which are homotopic among formal contact structures, and if our existence proof is smooth in families, we would obtain a family of contact structures $\xi_t$ interpolating between $\xi_0$ and $\xi_1$, by which we conclude that $\xi_0$ is contactomorphic to $\xi_1$. 
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Nevertheless, this approach is productive for a partial classification, those of overtwisted contact manifolds.

Theorem (Borman–Eliashberg–M., 2015)

Let $\eta$ be a formal contact structure on a smooth manifold $Y$. Then $\eta$ is realized by a unique overtwisted contact structure.
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This generalizes earlier work of [Eliashberg 1989], where the same result was established for 3–manifolds $Y$. 
Of course, this theorem is only interesting so far as we understand the definition of overtwistedness. (To get uniqueness within some subset of contact structures, we only need the axiom of choice.)
Overtwistedness

Of course, this theorem is only interesting so far as we understand the definition of overtwistedness. (To get uniqueness within some subset of contact structures, we only need the axiom of choice.) There are many equivalent characterizations of overtwistedness compatible with natural geometric decompositions and submanifolds in contact geometry [Casals–M.–Presas].
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There are many equivalent characterizations of overtwistedness compatible with natural geometric decompositions and submanifolds in contact geometry [Casals–M.–Presas]. Ultimately, the most essential property of overtwistedness is semi-locality: there is an overtwisted contact structure on the open ball, and any contact manifold containing an overtwisted open set is itself overtwisted.
Example: contact structures on $S^5$

Let $\Theta$ be the set of contact structures on $S^5$, up to diffeomorphism. A nice property of $S^5$ is that it admits a unique formal contact structure. $\Theta$ is a monoid, under contact connected sum. What do we know about this monoid?

- It has an identity element: if $\xi_{\text{std}} = \partial_{\infty} C^3$, then $\xi_{\text{std}} \# \xi \sim \xi$ for all $\xi \in \Theta$.

- It is countable (elementary), but not finitely generated [Ustilovsky 1999, Kwon–Van Koert 2016].

- It has a zero element: if $\xi_{\text{OT}}$ is the (unique) overtwisted contact structure, then $\xi_{\text{OT}} \# \xi = \xi_{\text{OT}}$ for all $\xi \in \Theta$. [BEM]

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New topic: submanifolds

The most important submanifolds in symplectic and contact geometry are, respectively, Lagrangian and Legendrian submanifolds.

\[ L \subseteq (X, \omega) \] is called Lagrangian if \( \omega|_L = 0 \) and \( \dim L = \frac{1}{2} \dim X \).

For Kähler manifolds this is equivalent to \( iTL = TL \perp \).

\[ \Lambda \subseteq (Y, \xi) \] is called Legendrian if \( T\Lambda \subseteq \xi \) and \( \dim \Lambda = \frac{1}{2} (\dim Y - 1) \).

If \( \Lambda \subseteq Y = X \cap S^{2N-1} \) for \( X \subseteq \mathbb{C}^N \), and \( L = (-\epsilon, \epsilon) \times \Lambda \subseteq X \) is defined by the gradient flow of \( |\cdot|^2 |_X \), then \( L \) is Lagrangian if and only if \( \Lambda \) is Legendrian.
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Yet again: topology vs. geometry

We can define formal Lagrangian and formal Legendrian submanifolds. These are smooth embeddings $L \subseteq X$ or $\Lambda \subseteq Y$ which are “homotopically framed” like Lagrangians/Legendrians.

Often the topology disagrees with the geometry: for instance there is a formal Lagrangian $S^3 \subseteq \mathbb{C}^3$, but there is no genuine Lagrangian in $\mathbb{C}^3$ diffeomorphic to $S^3$ [Gromov 1985].
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Loose Legendrians

When $\dim Y = 2n - 1 \geq 5$, every formal Legendrian is isotopic to a genuine Legendrian [Eliashberg 1990]. That is, for the existence problem of Legendrians in high dimensions, topology agrees with the geometry. This is false in dimension 3 [Bennequin 1982]. In all dimensions, uniqueness fails to hold: there are many distinct Legendrians isotopic as formal Legendrians. [Chekanov 2002, Ekholm–Etnyre–Sullivan 2005].

Theorem (M.)

Let $(Y, \xi)$ be a contact manifold of dimension at least 5. Then every formal Legendrian is isotopic to a unique loose Legendrian. Similar to overtwistedness for contact structures, the most important property defining loose Legendrians is that it is semi–local.
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Similar to overtwistedness for contact structures, the most important property defining loose Legendrians is that it is semi–local.
Application: construction of Lagrangians

Because of the tight relationship between Legendrians and Lagrangians, loose Legendrians can be used to construct Lagrangians.


Let $L$ be any closed 3–manifold, then there is a Lagrangian embedding $L \# S^2 \times S^1 \subseteq C^3$.

These Lagrangians are additionally interesting because they are "exotic" from the perspective of Lagrangian Floer theory: there are no pseudo-holomorphic curves passing through a generic point.
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Application: construction of Lagrangians

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Let $X$ be an formal Stein manifold. Then $X$ is equivalent to a unique flexible Stein structure.

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The Russell cubic threefold $X = \{x + x^2 y + w^3 + z^2 = 0\} \subseteq \mathbb{C}^4$ is symplectomorphic to $\mathbb{C}^3$.

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Trivializing a Koras–Russell cubic. (see [Casals–M.])