Smoothing finite group actions on three-manifolds

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Wild group actions

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Proposition

If $\dim M \leq 2$*, then every finite group action on* M *is tame.*

Wild group actions on three-manifolds

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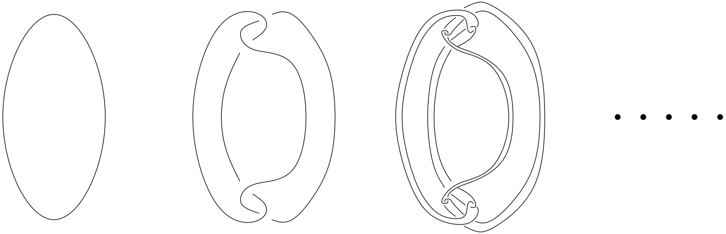
Corollary

There exists a wild action of $\mathbb{Z}/2$ on S^3.

Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be nested neighborhoods of Bing doubles $B_1, B_2, B_3, \ldots$



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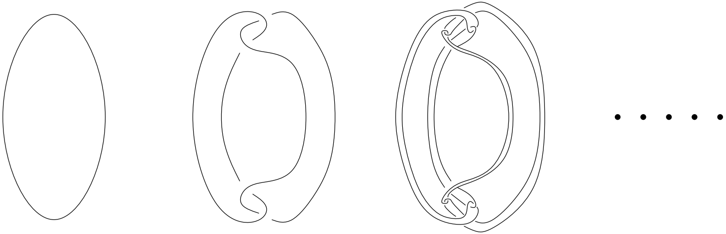

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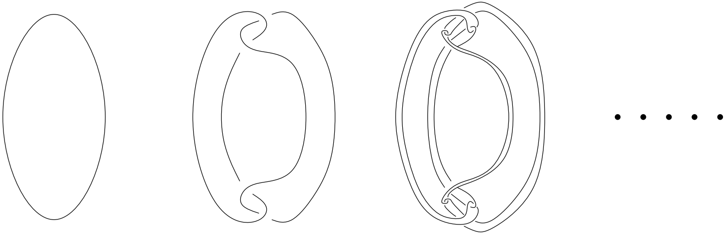

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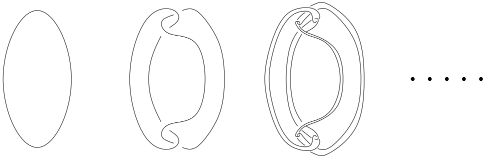
The double of the Alexander horned sphere is obtained from S^3 by collapsing the fibers of $C \times [0,1] \to C$:

$$X \cup_{S^2} X = S^3/\sim.$$



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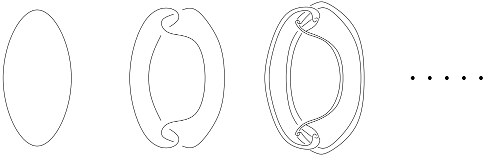
For every $\varepsilon > 0$*, there exists* $m < \infty$ *such that* N_m *can be isotoped inside* N_1 *so that every component has diameter* $\leq \varepsilon$*.*



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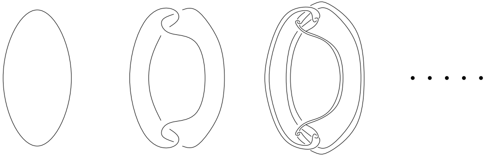

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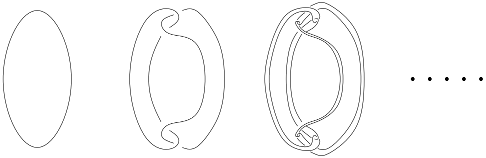
This reasoning moreover exhibits Bing's wild involution $\sigma : S^3 \to S^3$ as a uniform limit of smooth involutions.



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Later examples of wild finite group actions on three-manifolds given by Montgomery–Zippin (1954), Bing (1964), and Alford (1966). In some of these examples, the fixed set is a wild knot, rather than a wild surface.

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For the remainder of this talk, we shall present the proof.

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This gives at least a coarse local understanding of finite group actions on three-manifolds.

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Step 3 involves the *lattice of incompressible surfaces* introduced in the author's work on Hilbert–Smith conjecture in dimension three.

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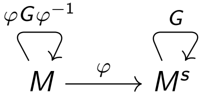
- Now $\varphi G \varphi^{-1} \curvearrowright M$ is smooth and approximates G.



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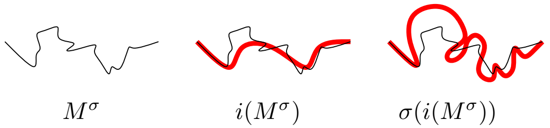

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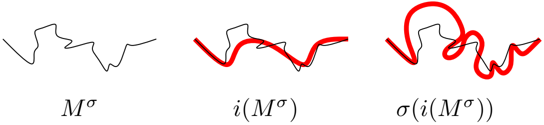

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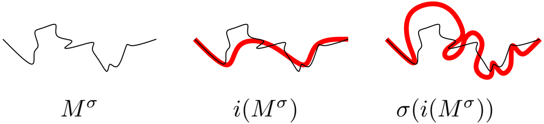

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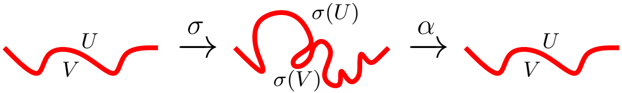
Now we define a new involution:

$$\tilde{\sigma} := \begin{cases} \alpha\sigma & U \\ (\alpha\sigma)^{-1} & V \end{cases}$$



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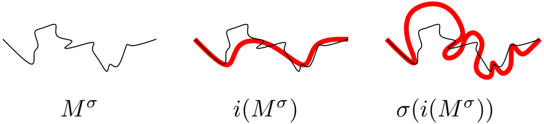

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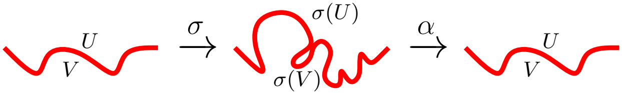

The fixed set of $\tilde\sigma$ is $i(M^\sigma)$ which is tame, and now $\tilde\sigma$ can be smoothed as in Step 1.

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If $x \in X$, then either:

- the stabilizer G_x has order 2 and $\dim_x M^{G_x} \in \{0, 1\}$ (so x is in the right hand side above), or
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In the latter case, consider the index ≤ 2 subgroup $G_x^\circ \leq G_x$ preserving orientation at x. Since $|G_x^\circ| > 1$, it has an element of prime order which preserves orientation at x. □

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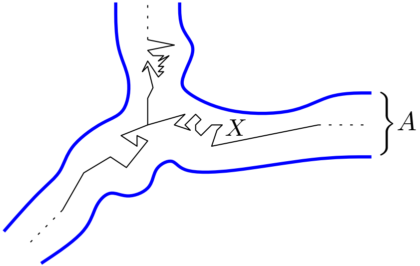
Fix a G-invariant neighborhood A with smooth boundary of X together with a finite G-equivariant open covering $A = \bigcup_i U_i$ with no triple overlaps (G is allowed to permute the U_i).



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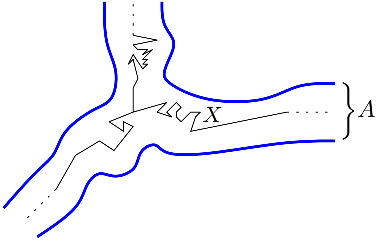

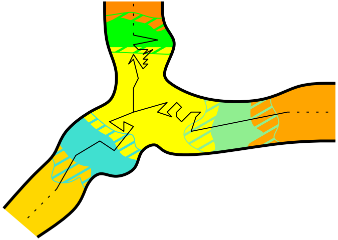

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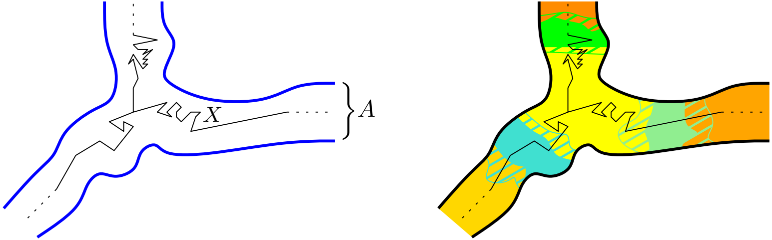

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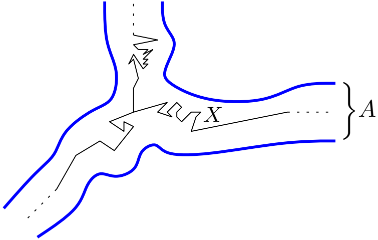

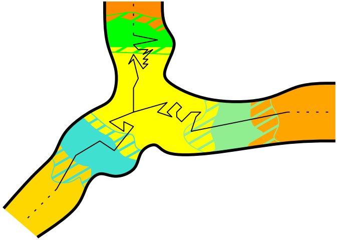

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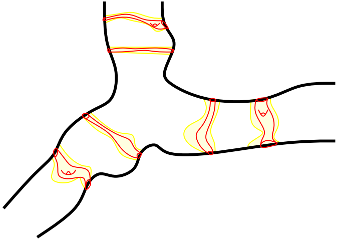

Step 3: Smoothing over the remainder of M

In each $U_i \cap U_j$, consider the set $\mathcal{S}(U_i \cap U_j)$ of isotopy classes of properly embedded surfaces $F \subseteq U_i \cap U_j$ such that:

- F is incompressible and ∂-incompressible.
- F minimally separates the U_i end from the U_j end.

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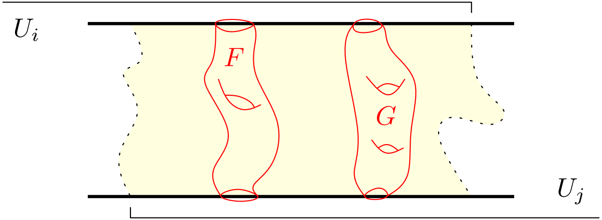

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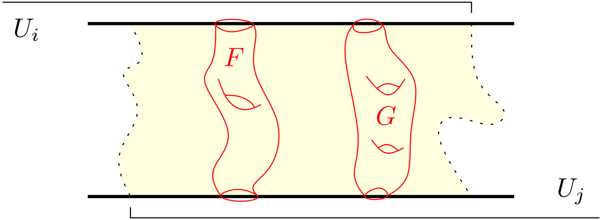

Lemma

$\mathcal{S}(U_i \cap U_j)$ is a poset, where $\mathfrak{F} \leq \mathfrak{G}$ iff there are representatives $F, G \subseteq U_i \cap U_j$ where F separates G from the U_i end.

Step 3: Smoothing over the remainder of M

Proposition

The poset $\mathcal{S}(U_i \cap U_j)$ is a lattice (every non-empty finite subset has a least upper bound).

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Every $\mathfrak{F} \in \mathcal{S}(U_i \cap U_j)$ has an area-minimizing representative, and work of Freedman–Hass–Scott (1983) implies that these representatives simultaneously realize all order relations in $\mathcal{S}(U_i \cap U_j)$.

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There exist $\mathfrak{F}_{ij} \in \mathcal{S}(U_i \cap U_j)$ which are G-invariant.

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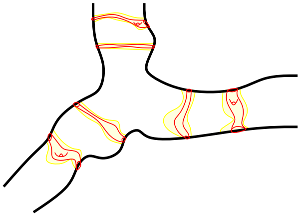
Corollary

There exist $\mathfrak{F}_{ij} \in \mathcal{S}(U_i \cap U_j)$ which are G-invariant.

Proof.

Start with arbitrary $\mathfrak{F}^0_{ij}$, take their orbit under the action of G, and let $\mathfrak{F}_{ij}$ be the least upper bound of this orbit. □

Step 3: Smoothing over the remainder of M



We may now conclude the proof:

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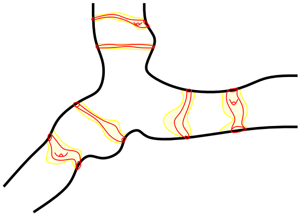

We may now conclude the proof:

- Cutting A along the surfaces F_{ij}, we obtain pieces A_i, and $G \curvearrowright \bigsqcup_i A_i$ (up to homotopy).

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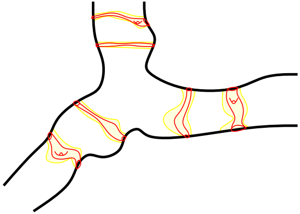

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- We upgrade this homotopy action to an action by diffeomorphisms by applying the JSJ decomposition to the pieces A_i and using Thurston's hyperbolization and Marden's rigidity results (among other major results in three-dimensional topology).

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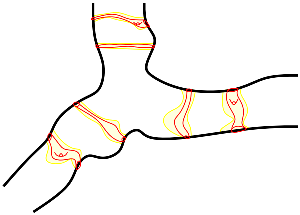

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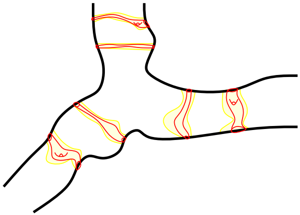

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The resulting smooth action of G on M can be made arbitrarily close to the original action in the uniform topology by taking the neighborhood A and the open sets U_i to be sufficiently small. □