Stable Homotopy Refinements and Khovanov homology

Robert Lipshitz\textsuperscript{1} and Sucharit Sarkar\textsuperscript{2}

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Special thanks to our collaborator Tyler Lawson, whose perspective is reflected throughout.

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\textsuperscript{2} SS was supported by NSF CAREER Grant DMS-1643401 and NSF FRG Grant DMS-1563615
Part 1: Stable homotopy refinements
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- Morse homology
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- General strategies for spatial refinements
- Flow categories and realization
Morse homology

\[ \chi(M) = \sum_{p \in \text{Crit}(f)} (-1)^{\text{ind}(p)} = (-1)^{\text{ind}(a)} + (-1)^{\text{ind}(b)} + (-1)^{\text{ind}(c)} + (-1)^{\text{ind}(d)} = 1 + (-1) + 1 + 1 = 2. \]

Categorify

\[ C_n(M; f) = \mathbb{Z} \langle p \in \text{Crit}(f) \mid \text{ind}(p) = n \rangle \]

\[ \partial : C_n(M; f) \to C_{n-1}(M; f) \]

\[ \partial(p) = \sum_{\text{ind}(q) = n-1} \# M(p, q) \]

signed count of flowlines of \(-\vec{\nabla} f\) from \(p\) to \(q\)
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Homology

\[ \mathbb{Z} \quad 0 \quad \mathbb{Z} \]

signed count of flowlines of \(-\nabla f\) from \(p\) to \(q\)
# Floer homology and categorification

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(and many others...)
The Cohen-Jones-Segal realization question

**Question.** (Cohen-Jones-Segal) Are these Floer homologies the homologies of naturally associated spaces?
Seems not have a natural cup product, so perhaps a spectrum (or, sometimes, pro-spectrum) instead of space?
The Cohen-Jones-Segal realization question

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**Spatial Refinement Problem.** Given a chain complex $C_*$ with distinguished basis, arising in an interesting way, construct a CW spectrum $X$ with $C_{\text{cell}}^*(X) \cong C_*$ with the distinguished basis given by the cells.
A theorem of Carlsson’s

**Question.** Is there a universal way of refining chain complexes, i.e.,

\[ C^* \text{cell} \]

\[ \text{CW spectra} \rightarrow \text{Chain complexes} \]

\[ \text{?} \]

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\[ C^\text{cell}_* \]

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\[ ? \]

**Theorem.** No.

**Proof.**

- (Carlsson ’81) Let \( G = \mathbb{Z}/2 \times \mathbb{Z}/2 \). There is a \( \mathbb{Z}[G] \)-module \( P \) which is not the homology of any \( G \)-equivariant (Moore) space.
- \( P \) is the homology of a chain complex over \( \mathbb{Z}[G] \).
Applications of spatial refinements

• Spectra have more information than chain complexes:
  • Steenrod operations on cohomology,
  • Homotopy groups, K-theory, . . .
• Maps between spectra have much more information than maps between groups.
• Even maps between spheres are interesting.
• For group actions on spaces, there are meaningful notions of fixed sets, and localization theorems on equivariant cohomology (Smith theory).
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General strategies for spatial refinements

- Cohen-Jones-Segal '95 gave a general procedure using higher-dimensional moduli spaces.
- Manolescu '03, Kronheimer-Manolescu used finite-dimensional approximation (following Furuta and Bauer) and the Conley index to refine Seiberg-Witten Floer homology.
- Kragh used finite-dimensional approximation (following Viterbo) to realize the Viterbo transfer for Lagrangians in cotangent bundles as a map of spectra.
- Hu-Kriz-Kriz '16, Lawson-Lipshitz-Sarkar used functors from the Burnside category to spaces to refine Khovanov homology. One could try to factor through other categories, as well.
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Flow categories and their realizations

A framed flow category is a way of encoding the moduli space of flows in Morse theory or Floer theory. Cohen-Jones-Segal turn a framed flow category into a CW spectrum.
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\[
\begin{array}{cccc}
\text{Object} & \text{Grading} \\
\hline
a & 0 \\
b & 1 \\
c & 2 \\
d & 2 \\
\end{array}
\]

Morphisms
\[\text{Hom}(c, b) = \{\alpha\}, \text{Hom}(d, b) = \{\beta\}, \text{Hom}(b, a) = \{\gamma, \delta\}\]
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**Morphisms**

Hom($c, b$) = $\{\alpha\}$, Hom($d, b$) = $\{\beta\}$, Hom($b, a$) = $\{\gamma, \delta\}$

Hom($c, a$) = $\langle\alpha, \delta\rangle$ $\langle\alpha, \gamma\rangle$
Flow categories and their realizations

A **framed flow category** is a way of encoding the moduli space of flows in Morse theory or Floer theory. Cohen-Jones-Segal turn a framed flow category into a CW spectrum.

A framed flow category consists of objects and morphisms. The objects are labeled as $a$, $b$, $c$, and $d$. The grading of each object is given in the table below:

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The morphisms between objects are described as follows:

- $\text{Hom}(c, b) = \{\alpha\}$
- $\text{Hom}(d, b) = \{\beta\}$
- $\text{Hom}(b, a) = \{\gamma, \delta\}$
- $\text{Hom}(c, a) = \{(\alpha, \delta), (\alpha, \gamma)\}$
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**Morphisms**

\[
\begin{align*}
\text{Hom}(c, b) &= \{\alpha\}, \quad \text{Hom}(d, b) = \{\beta\}, \quad \text{Hom}(b, a) = \{\gamma, \delta\} \\
\text{Hom}(c, a) &= (\alpha, \delta) (\alpha, \gamma), \quad \text{Hom}(d, a) = (\beta, \delta) (\beta, \gamma)
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(And some framing data.)
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(And some framing data.)

\[
\{\ast\} \amalg \bigoplus_{x \in \text{Ob}} D^{\text{gr}(x)+N} / \sim
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- $\text{Hom}(c, a) = (\alpha, \delta)$, $\text{Hom}(d, a) = (\beta, \delta)$

(And some framing data.)

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Morphisms

- $\text{Hom}(c, b) = \{\alpha\}$, $\text{Hom}(d, b) = \{\beta\}$, $\text{Hom}(b, a) = \{\gamma, \delta\}$
- $\text{Hom}(c, a) = (\alpha, \delta)$, $\text{Hom}(d, a) = (\beta, \delta)$

(And some framing data.)

$\bigl(\{\ast\} \amalg \amalg_{x \in \text{Ob}} D_{d, a}^{\text{gr}(x)+N}\bigr) / \sim$
Flow categories and their realizations

A framed flow category is a way of encoding the moduli space of flows in Morse theory or Floer theory. Cohen-Jones-Segal turn a framed flow category into a CW spectrum.

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(And some framing data.)

$$\{\ast\} \coprod \coprod_{x \in \text{Ob}} D^{\text{gr}(x)+N} / \sim \cong S^1 \vee S^3 = \Sigma(S^2_+).$$
Part 2: Khovanov homology and homotopy
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- State sums and the Jones polynomial
Part 2: Khovanov homology and homotopy

- State sums and the Jones polynomial
- The Khovanov cube
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- The Khovanov Burnside functor
- Extensions
- Applications
- Some open questions
State sums and the Jones polynomial

Knot $K$
State sums and the Jones polynomial

Knot $K$

Cube of resolutions (Kauffman '87)
State sums and the Jones polynomial

Knot $K$

Cube of resolutions (Kauffman ‘87)

State sum

$V_K(q) = \pm q^n \sum_{v \in \{0,1\}^c} (-q)^{|v|} (q + q^{-1})^{k(v)}$
State sums and the Jones polynomial

Knot $K$

Cube of resolutions (Kauffman '87)

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State sums and the Jones polynomial

Knot $K$

Cube of resolutions (Kauffman '87)

$\begin{align*}
\sum_{v \in \{0,1\}^c} & (-q)^{|v|} (q + q^{-1})^{k(v)} \\
& (-q)^1(q + q^{-1})^3 \\
& (-q)^2(q + q^{-1})^2 \\
& (-q)^0(q + q^{-1})^2 \\
& (-q)^1(q + q^{-1})^1 \\
\end{align*}$

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\[
(-q)^1(q + q^{-1})^3 \\
(-q)^2(q + q^{-1})^2 \\
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\]

State sum

\[
V_K(q) = \pm q^n \sum_{v \in \{0,1\}^c} (-q)^{|v|}(q + q^{-1})^{k(v)} = q + q^{-1}
\]
The Khovanov cube
(Khovanov '99)
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Khovanov Frobenius algebra

\[ V = \mathbb{Z}[x]/(x^2) \]
The Khovanov cube
(Khovanov '99)

Khovanov Frobenius algebra (1 + 1 TQFT)

\[ \text{circle} \rightarrow V = \mathbb{Z}[x]/(x^2) \]
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\[ \text{II} \rightarrow \otimes \]
The Khovanov cube
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- circle $\rightarrow$ $V = \mathbb{Z}[x]/(x^2)$
- $\mathbb{P} \rightarrow \otimes$
- merge $\rightarrow$ multiplication $m: V \otimes V \rightarrow V$
The Khovanov cube
(Khovanov '99)

Khovanov Frobenius algebra (1 + 1 TQFT)

- circle $\mapsto V = \mathbb{Z}[x]/(x^2)$
- $\mathbb{I} \mapsto \otimes$
- merge $\mapsto$ multiplication $m: V \otimes V \to V$
- split $\mapsto$ comultiplication $\Delta: V \to V \otimes V$
  - $1 \mapsto 1 \otimes x + x \otimes 1$
  - $x \mapsto x \otimes x$
The Khovanov cube
(Khovanov '99)

Khovanov Frobenius algebra \((1+1 \text{ TQFT})\)

- circle \(\longrightarrow V = \mathbb{Z}[x]/(x^2)\)
- II \(\longrightarrow \otimes\)
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- split \(\longrightarrow\) comultiplication \(\Delta: V \rightarrow V \otimes V\)

\[
\begin{align*}
1 & \mapsto 1 \otimes x + x \otimes 1 \\
 x & \mapsto x \otimes x
\end{align*}
\]
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  \(1 \mapsto 1 \otimes x + x \otimes 1\)
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TQFT

\[
\begin{align*}
V \otimes V & \xrightarrow{m \otimes \text{Id}} V \otimes V \\
\text{Id} \otimes \Delta & \downarrow \Delta \\
V \otimes V & \xrightarrow{m} V \\
\end{align*}
\]

Total complex

\[
\begin{align*}
V \otimes V & \xrightarrow{[-m \text{ Id} \otimes \Delta]} V \oplus (V \otimes V \otimes V) \\
& \xrightarrow{[m \otimes \text{Id}]} V \otimes V \\
\end{align*}
\]
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  \[
  1 \mapsto 1 \otimes x + x \otimes 1 \\
  x \mapsto x \otimes x
  \]

\[
\begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{m \otimes \text{Id}} & V \otimes V \\
\text{Id} \otimes \Delta & \uparrow & \Delta \\
V \otimes V & \xrightarrow{m} & V
\end{array}
\]

\[
V \otimes V \xrightarrow{[-m, \text{Id} \otimes \Delta]} V \oplus (V \otimes V \otimes V) \xrightarrow{[m \otimes \text{Id}]} V \otimes V
\]

Homology:

\[
\begin{array}{ccc}
0 & \mathbb{Z}^2 & 0
\end{array}
\]
Famous applications of Khovanov homology

\[ g_4(K) = 2(\mu - \lambda + 1) \]

• Example. \[ g_4(T_{p,q}) = g_3(T_{p,q}) = \mu(T_{p,q}) = (p-1)(q-1) \]

(Torus knot case conjectured by Milnor in '68, proved by Kronheimer and Mrowka in '93 using instanton gauge theory.)

Theorem. (Kronheimer-Mrowka '10) If \( \operatorname{rank}(\text{Kh}(K)) = 2 \), then \( K \) is the unknot.

• Proof uses instanton gauge theory.

Old conjecture. If \( \nu(K) = q + q - 1 \), then \( K \) is the unknot.

\[ u(T_{5,68}) = 134 \]
Famous applications of Khovanov homology

**Theorem.** (Rasmussen ’04) If $K$ is a positive knot, then

$$g_4(K) = g_3(K) = \frac{n - k + 1}{2}.$$  

- **Example.** $g_4(T_{p,q}) = g_3(T_{p,q}) = u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$. 

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Formal structure of Khovanov homotopy type

Link diagram $L$ \quad Finite CW spectrum $X^j_{Kh}(L), j \in \mathbb{Z}$ \quad Khovanov homology $Kh^{i,j}(L)$
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- **Cobordism** $L \rightarrow L'$ (movie presentation) → **Cellular map** $X^j_{Kh}(L) \leftarrow X^j_{Kh}(L')$ → **Usual cobordism map** $Kh^{i,j}(L) \rightarrow Kh^{i,j}(L')$

**Corollary.** (Lipshitz-Sarkar '12) There are Steenrod operations on Khovanov homology which are natural with respect to cobordism maps.

**Theorem.** (Lipshitz-Sarkar '12) There is an explicit combinatorial formula for the Steenrod square $Sq^2$:

$$Kh^{i,j}(K; \mathbb{Z}/2) \rightarrow Kh^{i+2,j}(K; \mathbb{Z}/2).$$

**Theorem.** (Seed '12) There are knots $K, K'$ with $Kh^{i,j}(K) \cong Kh^{i,j}(K')$ ($\forall i,j$) but $X^j_{Kh}(K) \not\simeq X^j_{Kh}(K').$

**Theorem.** (Lawson-Lipshitz-Sarkar '15) For any $k > 0$ there is a (non-prime) knot $K$ so that $Sq^k: Kh^{i,j}(K; \mathbb{Z}/2) \rightarrow Kh^{i+k,j}(K; \mathbb{Z}/2)$ is non-zero.
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The Burnside category

The Burnside category was used by Hu-Kriz-Kriz to refine Khovanov homology.

There are functors $B^P$,

$\text{Spectra} \to \text{X/Sets}/x \in X \text{S}$

Elmendorf-Mandell $K$-theory

cf. Barratt-Priddy-Quillen theorem

Can describe the map $B \to \text{Spectra}$ more explicitly, via Pontryagin-Thom construction (cf. Lawson-Lipshitz-Sarkar).

The Burnside category $B$ (of the trivial group) has:

- Objects finite sets $X$
- $\text{Hom}(X,Y)$ finite correspondences $X \rightarrow Y$
- Composition fiber products
- $2\text{Hom}(A,B)$ bijections $X \rightarrow Y \rightarrow A \rightarrow B$
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The *Burnside category* \( \mathcal{B} \) (of the trivial group) has:
- Objects finite sets \( X \)
- \( \text{Hom}(X,Y) \) finite correspondences

\[
\begin{array}{ccc}
A \\
\downarrow \\
X & \rightarrow & Y
\end{array}
\]

- Composition fiber products
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\[
\begin{array}{ccc}
\mathcal{B} & \longrightarrow & \text{Spectra} \\
X & \longrightarrow & \bigvee_{x \in X} S
\end{array}
\]

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- There are functors

\[
\begin{align*}
\mathcal{B} & \longrightarrow \mathcal{P}_{\text{ermu}} \longrightarrow \text{Spectra} \\
X & \longmapsto \text{Sets}/X \longmapsto \bigvee_{x \in X} S
\end{align*}
\]

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\[
\begin{align*}
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X & \longleftarrow \text{Sets}/X & \bigvee_{x \in X} & \mathcal{S}
\end{align*}
\]

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  - Composition fiber products
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Elmendorf-Mandell K-theory
cf. Barratt-Priddy-Quillen theorem
The Burnside category

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- There are functors:

  $\mathcal{B} \xrightarrow{\text{Perm}} \mathcal{P} \xrightarrow{\text{Sets}} \mathcal{Spectra}$

  $X \xrightarrow{\text{Sets}/X} \bigvee_{x \in X} S$

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Cube shaped diagrams

\[
\begin{array}{ccc}
C_{01} & \xrightarrow{f_1} & C_{11} \\
\uparrow f_0 & & \uparrow f_1 \\
C_{00} & \xrightarrow{f_0} & C_{10}
\end{array}
\]
Cube shaped diagrams

\[
\begin{array}{ccc}
C_{01} & \xrightarrow{f_{01}} & C_{11} \\
f_0 & \uparrow & \uparrow f_1 \\
C_{00} & \xrightarrow{f_{00}} & C_{10}
\end{array}
\]

Cone

\[
\begin{array}{ccc}
\text{Cone}(f_0) & \xrightarrow{f_0[1] \oplus f_{01}} & \text{Cone}(f_1) \\
\end{array}
\]

Cone

\[
\begin{array}{ccc}
\text{Cone}(f_0[1] \oplus f_{01})
\end{array}
\]

\[\sim=\]
Cube shaped diagrams

\[
\begin{align*}
C_{01} \xrightarrow{f_1} & C_{11} \\
C_{00} \xrightarrow{f_0} & C_{10}
\end{align*}
\]

\[
\begin{align*}
f_0 \uparrow & \quad \uparrow f_1 \\
C_{00} & \xrightarrow{f_0} C_{10}
\end{align*}
\]

\[
\begin{align*}
\text{Cone} & \quad \text{Cone}(f_0) \xrightarrow{f_0[1] \oplus f_1} \text{Cone}(f_1)
\end{align*}
\]

\[
\begin{align*}
\text{Cone} \quad \text{Cone}(f_{01}) \xrightarrow{f_{01}[1] \oplus f_1} \text{Cone}(f_{11})
\end{align*}
\]

\[
\begin{align*}
\text{Cone} \quad \text{Cone}(f_{00}[1] \oplus f_{11})
\end{align*}
\]

\[
\begin{align*}
Z_{01} & \leftarrow g_{11} \quad Z_{11} \\
Z_{00} & \leftarrow g_{00} \quad Z_{10}
\end{align*}
\]

\[
\begin{align*}
g_0 \downarrow & \quad \downarrow g_1 \\
Z_{00} & \xleftarrow{g_0} Z_{10}
\end{align*}
\]

\[
\begin{align*}
f_0 \uparrow & \quad \uparrow f_1 \\
C_{00} & \xrightarrow{f_0} C_{10}
\end{align*}
\]
Cube shaped diagrams

\[
\begin{array}{c}
C_{01} \xrightarrow{f_1} C_{11} \\
f_{0\bullet} \uparrow \quad \uparrow f_{1\bullet} \\
C_{00} \xrightarrow{f_{0\bullet}} C_{10}
\end{array}
\]

\[
\begin{array}{c}
C^* \\
\text{Cone}
\end{array}
\]

\[
\begin{array}{cc}
\text{Cone}(f_{0\bullet}) & \xrightarrow{f_{0\bullet}[1] \oplus f_{1\bullet}} \text{Cone}(f_{1\bullet}) \\
\text{C}^* & \text{C}_{\bullet}
\end{array}
\]

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\text{Cone}(g_{0\bullet}) & \xleftarrow{g_{0\bullet}[1]} \text{Cone}(g_{1\bullet}) \\
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Can totalize all at once, using homotopy colimits.
Works if diagrams are merely homotopy coherent.
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\begin{align*}
\text{Cone}(g_{0}) & \leftarrow \text{Cone}(g_{1}) \\
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\begin{align*}
Z_{01} & \xleftarrow{g_{01}} Z_{11} \\
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\end{align*}
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\end{align*}
\]

\[
\begin{align*}
C_{01} & \leftarrow Z_{01} \xleftarrow{g_1} Z_{11} \\
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\]

\[
\begin{align*}
\text{EM} & \quad \text{EM} \\
\end{align*}
\]

\[
\begin{align*}
X_{01} & \xleftarrow{A_1} X_{11} \\
A_{0} & \xleftarrow{A_0} X_{10} \\
X_{00} & \xleftarrow{A_0} X_{10} \\
\end{align*}
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\end{align*}
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- Can totalize all at once, using homotopy colimits.
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The Khovanov-Burnside functor
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Objects:
$$\text{Hom}(\pi_0(K_v), \{1, x\}) = \{1, x\}$$

$$\pi_0(K_u) \to \pi_0(K_v)$$

$$s^{-1}(y) \cap t^{-1}(z), y \in \{1, x\}, z \in \{1, x\}$$

Set of generators of the kernel of $$H_1(B) \to H_1(B \cap \{0, 1\} \times \mathbb{R})$$

2-morphisms: ...
The Khovanov-Burnside functor

- Objects: Hom(π₀(K⁺), {1, x}) = {1, x}π₀(K⁺)
The Khovanov-Burnside functor

- **Objects**: $\text{Hom}(\pi_0(K_v), \{1, x\}) = \{1, x\} \pi_0(K_v)$

- **Morphisms**: correspondence

- **2-morphisms**: ...
The Khovanov-Burnside functor

- Objects: $\text{Hom}(\pi_0(K_v), \{1, x\}) = \{1, x\} \pi_0(K_v)$

- Morphisms: correspondence $\{1, x\} \pi_0(K_u) \to \{1, x\} \pi_0(K_v)$
The Khovanov-Burnside functor

• Objects: \( \text{Hom}(\pi_0(K_v), \{1, x\}) = \{1, x\} \pi_0(K_v) \)

\[
\begin{array}{c}
1 \otimes 1 \\
1 \otimes x \\
x \otimes 1 \\
x \otimes x \\
\end{array}
\]

• Morphisms: correspondence \( \{1, x\} \pi_0(K_u) \rightarrow \{1, x\} \pi_0(K_v) \)

\[
s^{-1}(y) \cap t^{-1}(z), y \in \{1, x\} \pi_0(K_u), z \in \{1, x\} \pi_0(K_v)
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\[ s^{-1}(y) \cap t^{-1}(z), y \in \{1, x\}^{\pi_0(K_u)}, z \in \{1, x\}^{\pi_0(K_v)} \]

\[ \text{genus} = 0, |\{y(C) = 1\}| + |\{z(C) = x\}| = 1 \]

\[ \mapsto \{\text{pt}\} \]
The Khovanov-Burnside functor

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  \begin{bmatrix}
  1 \otimes 1 \\
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  \]

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  \]

  genus = 0, $|\{y(C) = 1\}| + |\{z(C) = x\}| = 1$

  $\mapsto \{\text{pt}\}$

  genus = 1, $|\{y(C) = 1\}| + |\{z(C) = x\}| = 0$

  Set of generators of the kernel of

  $\mapsto H^1(B) \rightarrow H^1(B \cap \{0, 1\} \times \mathbb{R})$
The Khovanov-Burnside functor

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  Purple curve \(\mapsto +1\)

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- genus = 0, $|\{y(C) = 1\}| + |\{z(C) = x\}| = 1$
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• 2-morphisms: $\ldots$

Purple curve $\mapsto +1$
Purple curve $\mapsto -1$
Extensions

• Lawson-Lipshitz-Sarkar: This definition of $\text{Kh}_{\mathcal{K}}$ due to Hu-Kriz-Kriz agrees with the original definition via flow categories due to Lipshitz-Sarkar.

• Jones-Lobb-Schütz, Lobb-Orson-Schütz: Many calculations, via moves to simplify flow categories.

• Lobb-Orson-Schütz, Willis: An extension to colored Khovanov homology.

• Jones-Lobb-Schütz: A conjectural extension to $\text{sl}_n$ Khovanov-Rozansky homology for certain kinds of knots ("matched diagrams").

• Sarkar-Scaduto-Stoffregen: An extension to Ozsváth-Rasmussen-Szabó's odd Khovanov homology.

• Borodzik-Politarczyk-Silvero, Musyt: Extension to periodic knots.
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Applications

• Lipshitz-Ng-Sarkar: A stable homotopy refinement of Plamenevskaya's transverse invariant, \( \Psi(K) \in \pi_0 s(\text{sl}(K)) \text{Kh}(K) \) = \( [\text{sl}(K) \text{Kh}(K), S] \).

• Lipshitz-Sarkar: A refinement of Rasmussen's \( s \)-invariant,

• \( s \text{Sq}_2(K) \in \{ s(K), s(K) + 2 \} \).

• \( 2g_4(K) \geq |s \text{Sq}_2(K)| \).

• Lawson-Lipshitz-Sarkar: For \( p, q > 0 \),

\[
2g_4(T_{p,q} \# 9_{42}) = 2g_4(T_{p,q}) + 2g_4(9_{42}) = (p-1)(q-1) + 1.
\]

\( 9_{42} \ni (3, 5) \) $\gi 9_{42} \ni (3, 5) \# 9_{42}$

• Feller-Lewark-Lobb: Call \( K \) squeezed if it is a slice of a minimal-genus cobordism from \( T_{p,q} \) to \( T_{p',q'} \).

• \( s \text{Sq}_2 \) gives one of the few known obstructions to \( K \) being squeezed.
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- Lipshitz-Ng-Sarkar: A stable homotopy refinement of Plamenevskaya’s transverse invariant,

\[ \Psi(K) \in \pi_0^s(X_{K_{sl}}^s(K)) = [X_{K_{sl}}^s(K), \mathbb{S}] \]
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Questions

• Are there refinements of Khovanov-Rozansky homologies? Khovanov-Rozansky’s HOMFLY-PT homology?
• Does $C_{\mathbb{P}^2}$ appear in the Khovanov spectrum of any link? More generally, do Chang spaces with no $\mathbb{Z}/2\mathbb{Z}$-summand in their cohomology appear?
• Is the Khovanov stable homotopy type natural under cobordisms? What about higher naturality for Khovanov chain complex or Khovanov homotopy type?
• Carry out Cohen-Jones-Segal’s program of refining Floer homology in general.
• Is there an intrinsic description of Floer (or Khovanov) stable homotopy types?
• (Many other open questions in the written version of this talk.)
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Thanks!

- We thank the many colleagues who have helped us learn this material, Mohammed Abouzaid, Ralph Cohen, Chris Douglas, Ciprian Manolescu, and many others...

- And most especially our collaborators on this project, Tyler Lawson and Lenhard Ng.

- Thanks also to the organizing committee for inviting us, our hosts for their hospitality, and all of you for listening.