Geometric structures and representations of discrete groups

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CNRS and IHÉS
General problem

Classify infinite discrete subgroups of $SL(n, \mathbb{R})$
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1. Overview: lattices and other discrete subgroups
2. Deformations
3. Important class: Anosov subgroups
4. Geometry: convex cocompactness
5. Beyond Anosov subgroups
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1. Overview: lattices and other discrete subgroups
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5. Beyond Anosov subgroups

Theorem (Danciger–Guéritaud–K. 2017; see also Zimmer 2017)

$P_1$-Anosov $\iff$ strong projective convex cocompactness
Section 1

Discrete subgroups of $\text{SL}(n, \mathbb{R})$: overview
Case $n = 2$

Γ discrete subgroup of $SL(2, \mathbb{R})$
Case $n = 2$

\[
\Gamma \text{ discrete subgroup of } \text{SL}(2, \mathbb{R}) \leadsto \mathbb{H}^2/\Gamma \text{ hyperbolic surface (orbifold)}
\]
Case $n = 2$

$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \rightsquigarrow \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold)

Two models of $\mathbb{H}^2$

Unit disc
\[
\{ z \in \mathbb{C} \mid |z| < 1 \}
\]

$z \mapsto i \frac{1+z}{1-z}$

Upper half plane
\[
\{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \]
Case \( n = 2 \)

\( \Gamma \) discrete subgroup of \( \text{SL}(2, \mathbb{R}) \) \( \hookrightarrow \mathbb{H}^2/\Gamma \) hyperbolic surface (orbifold)

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\( \text{SL}(2, \mathbb{R}) \) acts by Möbius transformations:
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}
\]
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**Uniformization Theorem (Poincaré, Koebe)**

*Any compact Riemann surface of genus $g \geq 2$ is conformally equivalent to $\mathbb{H}^2/\Gamma$ where $\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R})$.***

### Two models of $\mathbb{H}^2$

- **Unit disc**
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Any compact Riemann surface of genus \( g \geq 2 \) is conformally equivalent to \( \mathbb{H}^2/\Gamma \) where \( \Gamma \) discrete subgroup of \( \text{SL}(2, \mathbb{R}) \).

E.g. \( \Gamma \) generated by \( a_1, b_1, a_2, b_2 \)

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$\leadsto \mathbb{H}^2/\Gamma$

Such a group $\Gamma$ can be deformed inside $\text{SL}(2, \mathbb{R})$ while remaining discrete
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$\rightsquigarrow$ deformation space $\simeq \mathbb{R}^{6g-6}$ ("Teichmüller space")
Case $n = 2$

$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \cong \mathbb{H}^2 / \Gamma$ hyperbolic surface (orbifold)
Case $n = 2$

$\Gamma$ discrete subgroup of $SL(2, \mathbb{R}) \sim \mathbb{H}^2 / \Gamma$ hyperbolic surface (orbifold)
Case $n = 2$

Γ discrete subgroup of $\text{SL}(2, \mathbb{R}) \sim \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold)

Geometric classification
(“Fenchel–Nielsen coordinates”)

cusp

funnel
Case $n \geq 3$

Rigidity vs. deformability

Lattices of $G = \text{SL}(n, \mathbb{R})$

i.e. discrete subgroups $\Gamma$ with $\text{Haar}(G/\Gamma) < +\infty$

exist (Borel, Harish-Chandra)

▶ locally rigid (Selberg, Weil, Garland–Raghunathan):

all small deformations are trivial (conjugations)

▶ superrigid (Margulis):

can be realized in essentially only one way

"Smaller" discrete subgroups of $G$

▶ e.g. isomorphic to nonabelian free group or $\pi_1($surface$)$

▶ may admit nontrivial deformations, under which they may remain discrete

⇝ study such deformable discrete subgroups
Case \( n \geq 3 \)

Rigidity \quad vs. \quad deformability
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**Rigidity** vs. **deformability**

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“Smaller” discrete subgroups of \( G \)
- e.g. isomorphic to nonabelian free group or \( \pi_1(\text{surface}) \)
- may admit **nontrivial deformations**
Case \( n \geq 3 \)

**Rigidity vs. deformability**

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- may admit nontrivial deformations, under which they may remain discrete

~ study such deformable discrete subgroups
E.g. free group playing ping pong on $\mathbb{P}(\mathbb{R}^3)$ (Tits, ...)

$\begin{pmatrix} t \\ t^{-1} \end{pmatrix} \in \text{SL}(3, \mathbb{R})$, $t \gg 1$

Observation:
Any reduced word in $a$, $a^{-1}$, $b$, $b^{-1}$ sends the white region into the union of the four colored disks.

Consequence:
$\Gamma = \langle a, b \rangle$ nonabelian free group, discrete in $\text{SL}(3, \mathbb{R})$.

NB: $\Gamma$ admits nontrivial deformations, under which it remains discrete.
E.g. free group playing ping pong on $\mathbb{P}(\mathbb{R}^3)$ (Tits, ...)

$$a = \begin{pmatrix} t \\ 1 \\ t^{-1} \end{pmatrix} \in \text{SL}(3, \mathbb{R})$$
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$$a = \begin{pmatrix} t & 1 \\ t^{-1} & 1 \end{pmatrix} \in \text{SL}(3, \mathbb{R}), \ t \gg 1$$

$b$ conjugate of $a$ in $\text{SL}(3, \mathbb{R})$

“transverse”

$\mathbb{P}(\mathbb{R}^3)$

$\mathcal{G} = \langle a, b \rangle$ nonabelian free group, discrete in $\text{SL}(3, \mathbb{R})$

$\mathcal{G}$ admits nontrivial deformations, under which it remains discrete
E.g. free group playing ping pong on $\mathbb{P}(\mathbb{R}^3)$ (Tits, ...)

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$\Gamma = \langle a, b \rangle$ nonabelian free group, discrete in $\text{SL}(3, \mathbb{R})$
E.g. free group playing ping pong on $\mathbb{P}(\mathbb{R}^3)$ (Tits, ...)

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"transverse"

\[
x_a^0, x_a^+, x_b^-, x_b^+
\]

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Section 2

From groups to representations
Deformations of discrete subgroups of $G = \text{SL}(n, \mathbb{R})$
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For an abstract finitely generated group $\Gamma$, find and study

\[ \rho : \Gamma \to G \]

injective and discrete representations
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- open sets of injective and discrete representations inside $\text{Hom}(\Gamma, G)$
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open sets of injective and discrete representations inside $\text{Hom}(\Gamma, G)$

(often modulo conjugation by $G$).
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For an abstract finitely generated group $\Gamma$, find and study

$$\text{open sets of injective and discrete representations inside } \text{Hom}(\Gamma, G)$$

(often modulo conjugation by $G$).

Fundamental example:
$\Gamma = \pi_1(S)$ where $S$ closed surface of genus $\geq 2$

The Teichmüller space of $S$ is

\[
\{\text{injective and discrete representations } \Gamma \to \text{SL}(2, \mathbb{R})\}/\text{SL}^\pm(2, \mathbb{R}).
\]
Examples of open sets of injective and discrete representations

\[ \rho_0 : \Gamma = \pi_1(S) \to \text{inj} \to \text{disc}, \]

\[ \text{SL}(2, \mathbb{R}) \to \text{G} = \text{SL}(n, \mathbb{R}) , \]

then deform in \( \text{G} \).

Standard embedding

Irreducible embedding

Theorem (Choi–Goldman 1993, Labourie 2006)

The whole connected component of \( \rho_0 \) in \( \text{Hom}(\Gamma, \text{G}) \) consists of injective and discrete rep's.

\[ \text{Hitchin'92} \]


\[ \Rightarrow \text{modulo conjugation}, \approx \mathbb{R} (n^2 - 1)(2g - 2) \]
Examples of open sets of injective and discrete representations

\[ \rho_0 : \Gamma = \pi_1(S) \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{embedding}} G = \text{SL}(n, \mathbb{R}), \text{ then deform in } G \]
Examples of open sets of injective and discrete representations

\[ \rho_0 : \Gamma = \pi_1(S) \overset{\text{inj.}}{\rightarrow} \overset{\text{disc.}}{\text{SL}(2, \mathbb{R})} \overset{\text{embedding}}{\rightarrow} G = \text{SL}(n, \mathbb{R}) \], then deform in \( G \)

**Standard embedding**

\[
\begin{pmatrix}
\text{SL}(2, \mathbb{R}) & 0 \\
0 & \text{Id}_{n-2}
\end{pmatrix}
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Examples of open sets of injective and discrete representations

\[ \rho_0 : \Gamma = \pi_1(S) \overset{\text{inj.}}{\longrightarrow} \text{SL}(2, \mathbb{R}) \overset{\text{embedding}}{\longrightarrow} G = \text{SL}(n, \mathbb{R}) , \text{ then deform in } G \]

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**Observation**

\[ \exists \text{ neighborhood of } \rho_0 \text{ in } \text{Hom}(\Gamma, G) \text{ consisting entirely of injective and discrete rep's.} \]
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The whole connected component of \( \rho_0 \) in \( \text{Hom}(\Gamma, G) \) consists of injective and discrete rep’s.

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The whole connected component of \( \rho_0 \) in \( \text{Hom}(\Gamma, G) \) consists of injective and discrete rep’s ("Hitchin representations").

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\[ \sim \text{modulo conjugation, } \sim \mathbb{R}^{(n^2-1)(2g-2)} \] (Hitchin’92)

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Examples of open sets of injective and discrete representations

\[ \rho_0 : \Gamma = \pi_1(S) \overset{\text{inj.}}{\to} \text{SL}(2, \mathbb{R}) \overset{\text{embedding}}{\to} G = \text{SL}(n, \mathbb{R}), \text{ then deform in } G \]

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"Higher Teichmüller space"

Theorem (Choi–Goldman 1993, Labourie 2006)

Any Hitchin representation \( \rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R}) \) is injective and discrete.

= continuous deformation of \( \rho_0 : \Gamma \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irreducible}} \text{SL}(n, \mathbb{R}) \)
Theorem (Choi–Goldman 1993 for $n = 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R})$ is injective and discrete.

= continuous deformation of $\rho_0 : \Gamma \xrightarrow{\text{inj./disc.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irreducible}} \text{SL}(n, \mathbb{R})$

Proof for $n = 3$: geometry
Theorem (Choi–Goldman 1993 for $n = 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R})$ is injective and discrete.

$\Rightarrow$ continuous deformation of $\rho_0 : \Gamma \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irreducible}} \text{SL}(n, \mathbb{R})$

Proof for $n = 3$: geometry

$\rho_0(\Gamma) \subset \text{SO}(2, 1)$
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Proof for $n = 3$: geometry

$\rho_0(\Gamma) \subset \text{SO}(2, 1) \Rightarrow$ preserves

$\{[x] \in \mathbb{P}(<0) | x_1^2 + x_2^2 - x_3^2 < 0\}$
Theorem (Choi–Goldman 1993 for $n = 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R})$ is injective and discrete.

\[ \begin{align*}
\text{continuous deformation of } \rho_0 : \Gamma \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xleftarrow{\text{irreducible}} \text{SL}(n, \mathbb{R})
\end{align*} \]

Proof for $n = 3$: geometry

\[ \rho_0(\Gamma) \subset \text{SO}(2, 1) \Rightarrow \text{preserves } \{ [x] \in \mathbb{P}(\mathbb{R}^3) | x_1^2 + x_2^2 - x_3^2 < 0 \} \]

\[ \rho \text{ preserves a properly convex open set in } \mathbb{P}(\mathbb{R}^3) \]
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Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R})$ is injective and discrete.

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Proof for $n = 3$: geometry

$\rho_0(\Gamma) \subset \text{SO}(2, 1) \Rightarrow$ preserves $\{[x] \in \mathbb{P}(\mathbb{R}^3) | x_1^2 + x_2^2 - x_3^2 < 0\}$

$\rho$ preserves a properly convex open set in $\mathbb{P}(\mathbb{R}^3)$ (convex and bounded in some affine chart)
Theorem (Labourie 2006 for general $n \geq 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R})$ is injective and discrete.

= continuous deformation of $\rho_0 : \Gamma \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irreducible}} \text{SL}(n, \mathbb{R})$

\[\text{Proof for } n \geq 3: \text{ dynamics}\]
**Theorem (Labourie 2006 for general \( n \geq 3 \))**

Any Hitchin representation \( \rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R}) \) is injective and discrete.

= continuous deformation of \( \rho_0 : \Gamma \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irreducible}} \text{SL}(n, \mathbb{R}) \)

**Proof for \( n \geq 3 \): dynamics**

- Key notion: boundary of \( \Gamma \)
  \( \partial_\infty \Gamma = \) visual boundary of a metric space \((X, d)\) on which \( \Gamma \) acts geometrically
Theorem (Labourie 2006 for general $n \geq 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \text{SL}(n, \mathbb{R})$ is injective and discrete.

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- Key notion: boundary of $\Gamma = \pi_1(S)$
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Theorem (Labourie 2006 for general $n \geq 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \to \SL(n, \mathbb{R})$ is injective and discrete.

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Proof for $n \geq 3$: dynamics

- Key notion: boundary of $\Gamma = \pi_1(S) \sim \partial_\infty \Gamma = \partial \tilde{S}$ (circle)
  \[ \partial_\infty \Gamma = \text{visual boundary of a metric space } (X, d) \text{ on which } \Gamma \text{ acts geometrically} \]

surface $S$

universal cover $\tilde{S} \simeq \mathbb{H}^2$
Theorem (Labourie 2006 for general $n \geq 3$)

Any Hitchin representation $\rho : \Gamma = \pi_1(S) \rightarrow \text{SL}(n, \mathbb{R})$ is injective and discrete.

= continuous deformation of $\rho_0 : \Gamma \xrightarrow{\text{inj.}} \text{SL}(2, \mathbb{R}) \xrightarrow{\text{irreducible}} \text{SL}(n, \mathbb{R})$

Proof for $n \geq 3$: dynamics

- Key notion: boundary of $\Gamma = \pi_1(S) \sim \partial_\infty \Gamma = \partial \tilde{S}$ (circle)
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Proof for $n \geq 3$: dynamics

- Key notion: boundary of $\Gamma = \pi_1(S) \sim \partial_\infty \Gamma = \partial S$ (circle)
  \( \partial_\infty \Gamma = \) visual boundary of a metric space $X, d$ on which $\Gamma$ acts geometrically

- Proof: For $\rho : \Gamma \to \text{SL}(n, \mathbb{R})$ Hitchin, show there exists a “boundary map”

\[
\xi : \partial_\infty \Gamma \to \text{Flag}(\mathbb{R}^n)
\]

\[
z \mapsto (\xi_1(z) \subset \xi_2(z) \subset \cdots \subset \xi_{n-1}(z))
\]

continuous, injective, compatible with actions $\Gamma \circ \partial_\infty \Gamma$ and $\Gamma \circ \text{Flag}(\mathbb{R}^n)$.
Boundary maps $\xi = (\xi_1, \ldots, \xi_{n-1}) : \partial_\infty \Gamma \to \text{Flag}(\mathbb{R}^n)$ for Hitchin rep’s

$n = 3$

$\xi_1(z)$

$\xi_2(z)$

$\text{Im}(\xi_1) = \text{boundary of convex set}$

$\xi_2(z) = \text{proj. line tangent to convex set at } \xi_1(z)$
Boundary maps $\xi = (\xi_1, \ldots, \xi_{n-1}) : \partial_\infty \Gamma \to \text{Flag}(\mathbb{R}^n)$ for Hitchin rep's

For $n = 3$:
- $\xi_1(z)$
- $\xi_2(z)$

$\text{Im}(\xi_1) =$ boundary of convex set
$\xi_2(z) =$ proj. line tangent to convex set at $\xi_1(z)$

For $n = 4$:
- $\xi_1(z)$
- $\xi_2(z)$
- $\xi_3(z)$

$\text{Im}(\xi_1) =$ $C^1$ curve, nontrivial homotopy (for $\rho_0$: "twisted cubic", equation $(t, t^2, t^3)$)
$\xi_2(z) =$ osculating line at $\xi_1(z)$
$\xi_3(z) =$ osculating plane at $\xi_1(z)$
Section 3

Anosov representations
Section 3

Anosov representations

- injective and discrete representations, forming open sets
Section 3

Anosov representations

- injective and discrete representations, forming open sets
- images of Anosov rep’s (“Anosov subgroups”) are some of the few reasonably understood discrete subgroups of $\text{SL}(n, \mathbb{R})$ beyond lattices
Finitely generated group $\Gamma = \pi_1(S)$ or free group or...

(Gromov hyperbolic group)

Boundary $\partial_\infty \Gamma =$ circle or Cantor set or...

Choose $1 \leq i \leq n/2$.

Definition (Labourie 2006, Guichard–Wienhard 2012)

A representation $\rho : \Gamma \to \text{SL}(n, \mathbb{R})$ is $\mathbb{P}i$-Anosov if

$\exists \xi = (\xi_i, \xi_{n-i}) : \partial_\infty \Gamma \to \text{Flag}_i, n-i(\mathbb{R}^n) = \{ (V_i \subset V_{n-i}) | \text{dim} V = \}$

$\Gamma$-equivariant, continuous, injective, satisfying:

1. transversality ($\forall z \neq z' \in \partial_\infty \Gamma$,

$\xi_i(z) \oplus \xi_{n-i}(z') = \mathbb{R}^n$)

2. uniform contraction/expansion condition for some flow

"the intrinsic dynamics of $\Gamma$ on $\partial_\infty \Gamma$ are reflected in $\text{Flag}_i, n-i(\mathbb{R}^n)$ via $\xi"$

Important properties (Labourie):

$\triangleright$ Anosov representations are injective and discrete

$\triangleright$ the set of Anosov representations is open in $\text{Hom}(\Gamma, \text{SL}(n, \mathbb{R}))$
Finitely generated group $\Gamma = \pi_1(S)$ or free group or... (Gromov hyperbolic group)

Boundary $\partial_\infty \Gamma = \text{circle or Cantor set or...}$
Finitely generated group $\Gamma = \pi_1(S)$ or free group or... (Gromov hyperbolic group)
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**Definition (Labourie 2006, Guichard–Wienhard 2012)**

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\]

$\Gamma$-equivariant, continuous, injective, satisfying:

1. **transversality** ($\forall z \neq z' \text{ in } \partial_\infty \Gamma, \xi_i(z) \oplus \xi_{n-i}(z') = \mathbb{R}^n$)
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Finitely generated group $\Gamma = \pi_1(S)$ or free group or... (Gromov hyperbolic group)
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2. uniform contraction/expansion condition for some flow

**Theorem (Kapovich-Leeb-Porti, Guéritaud-Guichard-K.-Wienhard, 2014–15)**

Can replace 2 by

$$2', \quad \frac{i\text{-th singular value}}{(i+1)\text{-th singular value}}(\rho(\gamma)) \to \infty \quad \text{or} \quad 2'' \quad \frac{i\text{-th } |\text{eigenvalue}|}{(i+1)\text{-th } |\text{eigenvalue}|}(\rho(\gamma)) \to \infty$$
Finitely generated group $\Gamma = \pi_1(S)$ or free group or... (Gromov hyperbolic group) 
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### Definition (Labourie 2006, Guichard-Wienhard 2012)

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### Theorem (Kapovich-Leeb-Porti, Guéritaud-Guichard-K.-Wienhard, 2014–15)

*Can replace 2 by*

$$2', \quad \text{i-th singular value} \quad \frac{\rho(\gamma)}{(i+1)\text{-th singular value}} \xrightarrow{\gamma \to \infty} \infty \quad \text{or} \quad 2'' \quad \text{i-th \mid eigenvalue\mid} \quad \frac{\rho(\gamma)}{(i+1)\text{-th \mid eigenvalue\mid}} \xrightarrow{[\gamma] \to \infty} \infty$$
Finitely generated group $\Gamma = \pi_1(S)$ or free group or... (Gromov hyperbolic group)
Boundary $\partial_\infty \Gamma =$ circle or Cantor set or...
Choose $1 \leq i \leq n/2$.

**Definition (Labourie 2006, Guichard–Wienhard 2012)**

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$\Gamma$-equivariant, continuous, injective, satisfying:

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“the intrinsic dynamics of $\Gamma$ on $\partial_\infty \Gamma$ are reflected in $\text{Flag}_{i,n-i}(\mathbb{R}^n)$ via $\xi$”

**Important properties (Labourie):**

- Anosov representations are **injective and discrete**
- the set of Anosov representations is open in $\text{Hom}(\Gamma, \text{SL}(n, \mathbb{R}))$
Examples

\( P_i \text{-Anosov representation } \rho : \Gamma \to \text{SL}(n, \mathbb{R}) \leadsto \text{boundary map } \xi : \partial_\infty \Gamma \to \text{Flag}_{i,n-i}(\mathbb{R}^n) \)
Examples

$P_i$-Anosov representation $\rho : \Gamma \to \text{SL}(n, \mathbb{R}) \leadsto$ boundary map $\xi : \partial_\infty \Gamma \to \text{Flag}_{i,n-i}(\mathbb{R}^n)$

- Hitchin representations are $P_i$-Anosov for all $i$ (Labourie)
Examples

\[ P_i\text{-Anosov representation } \rho : \Gamma \to \text{SL}(n, \mathbb{R}) \rightsquigarrow \text{boundary map } \xi : \partial_\infty \Gamma \to \text{Flag}_{i,n-i}(\mathbb{R}^n) \]

- Hitchin representations are \( P_i\)-Anosov for all \( i \) \((\text{Labourie})\)

- Small deformations of \( \pi_1(S) \hookrightarrow \text{SL}(2, \mathbb{R}) \overset{\text{standard}}{\twoheadrightarrow} \text{SL}(n, \mathbb{R}) \) are \( P_1\)-Anosov
Examples

$P_i$-Anosov representation $\rho : \Gamma \to \text{SL}(n, \mathbb{R}) \rightsquigarrow \text{boundary map } \xi : \partial_\infty \Gamma \to \text{Flag}_{i,n-i}(\mathbb{R}^n)$

- Hitchin representations are $P_i$-Anosov for all $i$ (Labourie)

- Small deformations of $\pi_1(S) \hookrightarrow \text{SL}(2, \mathbb{R}) \overset{\text{standard}}{\longrightarrow} \text{SL}(n, \mathbb{R})$ are $P_1$-Anosov

- Free groups playing ping pong on $\mathbb{P}(\mathbb{R}^n)$ are $P_1$-Anosov

Geometric interpretation for Anosov representations?


$\text{Im}(\xi_1) \subset \mathbb{P}(\mathbb{R}^3)$
Examples

- Hitchin representations are $P_i$-Anosov for all $i$ (Labourie)
- Small deformations of
  $\pi_1(S) \hookrightarrow \text{SL}(2, \mathbb{R})$ \text{standard} $\rightarrow \text{SL}(n, \mathbb{R})$
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- Free groups playing ping pong on $\mathbb{P}(\mathbb{R}^n)$ are $P_1$-Anosov

Geometric interpretation for Anosov representations?

$P_i$-Anosov representation $\rho : \Gamma \rightarrow \text{SL}(n, \mathbb{R}) \rightsquigarrow$ boundary map $\xi : \partial_\infty \Gamma \rightarrow \text{Flag}_{i,n-i}(\mathbb{R}^n)$


**Examples**

\[ P_i \text{-Anosov representation } \rho : \Gamma \to \text{SL}(n, \mathbb{R}) \leadsto \text{boundary map } \xi : \partial_\infty \Gamma \to \text{Flag}_{i,n-i}(\mathbb{R}^n) \]

- Hitchin representations are \( P_i \)-Anosov for all \( i \) \text{ (Labourie)}

- Small deformations of \( \pi_1(S) \hookrightarrow \text{SL}(2, \mathbb{R}) \xrightarrow{\text{standard}} \text{SL}(n, \mathbb{R}) \) are \( P_1 \)-Anosov

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**Geometric interpretation for Anosov representations?**

(Choi-Goldman, Frances, Barbot, Guichard-Wienhard, Kapovich-Leeb-Porti, Guéritaud-Guichard-K.-Wienhard, Collier-Tholozan-Toulisse, ...)}
Section 4

Convex cocompactness
Classical setting: $\text{SL}(2, \mathbb{R})$
Classical setting: $\text{SL}(2, \mathbb{R})$

Recall:
$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \rightsquigarrow \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold)
(finitely generated)
Classical setting: $SL(2, \mathbb{R})$

Recall:
\[ \Gamma \text{ discrete subgroup of } SL(2, \mathbb{R}) \rightsquigarrow \mathbb{H}^2 / \Gamma \text{ hyperbolic surface (orbifold)} \]
(finitely generated)
Classical setting: $\text{SL}(2, \mathbb{R})$

Recall: 
$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \rightarrow \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold) 
(finitely generated)

**Definition**

$\Gamma$ is **convex cocompact** if $\mathbb{H}^2/\Gamma$ has no cusp

---

$\mathbb{H}^2/\Gamma$ has no cusp

Funnel
Classical setting: \( SL(2, \mathbb{R}) \)

Recall:
\[ \Gamma \text{ discrete subgroup of } SL(2, \mathbb{R}) \rightarrow \mathbb{H}^2/\Gamma \text{ hyperbolic surface (orbifold)} \]
(finitely generated)

**Definition**
\( \Gamma \) is **convex cocompact** if \( \mathbb{H}^2/\Gamma \) has no cusp, or equivalently if
\[ \exists \mathcal{C} \subset \mathbb{H}^2 \text{ convex, } \Gamma \text{-invariant with } \mathcal{C}/\Gamma \text{ compact } \neq \emptyset \]
Classical setting: $\text{SL}(2, \mathbb{R})$

Recall:
$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \twoheadrightarrow \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold)
(finitely generated)

**Definition**

$\Gamma$ is **convex cocompact** if $\mathbb{H}^2/\Gamma$ has no cusp, or equivalently if
$\exists \ C \subset \mathbb{H}^2$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$

Example:

$\Gamma = \langle \gamma_1, \gamma_2 \rangle$
Classical setting: $\text{SL}(2, \mathbb{R})$

Recall:
$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \leadsto \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold)
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$\Gamma$ is convex cocompact if $\mathbb{H}^2/\Gamma$ has no cusp, or equivalently if $\exists C \subset \mathbb{H}^2$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$

Example:
$\Gamma = \langle \gamma_1, \gamma_2 \rangle$

Fact
$\Gamma < \text{SL}(2, \mathbb{R})$ is Anosov $\iff$ $\Gamma$ is convex cocompact

Example for $\text{SL}(n, \mathbb{R}), n \geq 3$:
$\Gamma = \langle \gamma_1, \gamma_2 \rangle$
Classical setting: $\text{SL}(2, \mathbb{R})$

Recall:
$\Gamma$ discrete subgroup of $\text{SL}(2, \mathbb{R}) \leadsto \mathbb{H}^2/\Gamma$ hyperbolic surface (orbifold)
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$\Gamma$ is **convex cocompact** if $\mathbb{H}^2/\Gamma$ has no cusp, or equivalently if $\exists \mathcal{C} \subset \mathbb{H}^2$ convex, $\Gamma$-invariant with $\mathcal{C}/\Gamma$ compact $\neq \emptyset$

**Fact**
$\Gamma < \text{SL}(2, \mathbb{R})$ is **Anosov**
$\iff \Gamma$ is convex cocompact

Example:
$\Gamma = \langle \gamma_1, \gamma_2 \rangle$

$\mathbb{H}^2$

$\mathcal{C}$ convex

$\mathcal{C}/\Gamma$ compact

$\mathbb{H}^2/\Gamma$
Classical setting: $\text{SL}(2, \mathbb{R})$

Recall:
\[ \Gamma \text{ discrete subgroup of } \text{SL}(2, \mathbb{R}) \leadsto \mathbb{H}^2/\Gamma \text{ hyperbolic surface (orbifold)} \]
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**Definition**
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**Fact**
\[ \Gamma < \text{SL}(2, \mathbb{R}) \text{ is Anosov } \iff \Gamma \text{ is convex cocompact} \]

Example:
\[ \Gamma = \langle \gamma_1, \gamma_2 \rangle \]

What about $\text{SL}(n, \mathbb{R}), \ n \geq 3$?
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

Definition

$\Gamma$ discrete $\subset \text{SL}(2, \mathbb{R})$ is **convex cocompact** if $\exists \ C \subset \mathbb{H}^2$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$.
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \cong \text{SL}(2, \mathbb{R})/\text{SO}(2) \rightsquigarrow$ generalize to $X_n = \text{SL}(n, \mathbb{R})/\text{SO}(n)$ symmetric space

**Definition**

$\Gamma$ discrete $< \text{SL}(2, \mathbb{R})$ is **convex cocompact** if $\exists C \subset \mathbb{H}^2$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \cong \text{SL}(2, \mathbb{R})/\text{SO}(2) \leadsto$ generalize to $X_n = \text{SL}(n, \mathbb{R})/\text{SO}(n)$ symmetric space

**Definition**

$\Gamma$ discrete $< \text{SL}(n, \mathbb{R})$ is **convex cocompact in** $X_n$ if $\exists \, C \subset X_n$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$.
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \simeq \text{SL}(2, \mathbb{R})/\text{SO}(2) \sim \text{generalize to } X_n = \text{SL}(n, \mathbb{R})/\text{SO}(n)$ symmetric space

**Definition**

\[ \Gamma \text{ discrete } < \text{SL}(n, \mathbb{R}) \text{ is convex cocompact in } X_n \text{ if } \exists C \subset X_n \text{ convex, } \Gamma\text{-invariant with } C/\Gamma \text{ compact } \neq \emptyset \]

- For $n \geq 3$, the vast majority of Anosov subgroups of $\text{SL}(n, \mathbb{R})$ are not convex cocompact in $X_n$ (Kleiner-Leeb’06, Quint’05)
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \simeq \text{SL}(2, \mathbb{R})/\text{SO}(2) \rightsquigarrow$ generalize to $X_n = \text{SL}(n, \mathbb{R})/\text{SO}(n)$ symmetric space

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- Yet they have a number of dynamical and topological properties that nicely generalize those of convex cocompact subgroups of $\text{SL}(2, \mathbb{R})$ (Labourie’06, Guichard-Wienhard’12, Kapovich-Leeb-Porti’14-18, ...)

Other form of convex cocompactness?
First attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \cong \text{SL}(2, \mathbb{R})/\text{SO}(2)$ generalize to $X_n = \text{SL}(n, \mathbb{R})/\text{SO}(n)$ symmetric space

**Definition**

$\Gamma$ discrete $<$ $\text{SL}(n, \mathbb{R})$ is **convex cocompact in** $X_n$ if $\exists C \subset X_n$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$

- For $n \geq 3$, the vast majority of **Anosov** subgroups of $\text{SL}(n, \mathbb{R})$ are **not** convex cocompact in $X_n$ (Kleiner–Leeb’06, Quint’05)

- Yet they have a number of **dynamical and topological properties** that nicely generalize those of convex cocompact subgroups of $\text{SL}(2, \mathbb{R})$ (Labourie’06, Guichard–Wienhard’12, Kapovich–Leeb–Porti’14-18, ...)

$\leadsto$ other form of convex cocompactness?
Second attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$$\mathbb{H}^2 \cong \{ [x] \in \mathbb{P}(\mathbb{R}^3) \mid x_1^2 + x_2^2 - x_3^2 < 0 \}$$
Second attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \simeq \{ [x] \in \mathbb{P}(\mathbb{R}^3) \mid x_1^2 + x_2^2 - x_3^2 < 0 \} \leadsto \text{generalize to } \Omega \text{ properly convex open } \subset \mathbb{P}(\mathbb{R}^n)$
Second attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \simeq \{ [x] \in \mathbb{P}(\mathbb{R}^3) | x_1^2 + x_2^2 - x_3^2 < 0 \} \rightsquigarrow$ generalize to $\Omega$ properly convex open $\subset \mathbb{P}(\mathbb{R}^n)$

e.g.

$n = 3$
Second attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

$\mathbb{H}^2 \simeq \{ [x] \in \mathbb{P}(\mathbb{R}^3) \mid x_1^2 + x_2^2 - x_3^2 < 0 \} \leadsto$ generalize to $\Omega$ properly convex open $\subset \mathbb{P}(\mathbb{R}^n)$

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$\mathbb{H}^2 \cong \{[x] \in \mathbb{P}(\mathbb{R}^3) \mid x_1^2 + x_2^2 - x_3^2 < 0\} \rightsquigarrow$ generalize to $\Omega$ properly convex open $\subset \mathbb{P}(\mathbb{R}^n)$

**Definition**

$\Gamma$ discrete $\leq \text{SL}(n, \mathbb{R})$ is **convex cocompact** in $\mathbb{P}(\mathbb{R}^n)$ if there exist:

- $\Omega$ properly convex, $\Gamma$-invariant open $\subset \mathbb{P}(\mathbb{R}^n)$, and
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Second attempt: convex cocompactness for $\text{SL}(n, \mathbb{R})$

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(Case $\Omega = C$: “divisible convex sets”, see Benoist’00-06)
Second attempt: convex cocompactness for \( SL(n, \mathbb{R}) \)

\[
\mathbb{H}^2 \simeq \{ [x] \in \mathbb{P}(\mathbb{R}^3) \mid x_1^2 + x_2^2 - x_3^2 < 0 \} \leadsto \text{generalize to } \Omega \text{ properly convex open } \subset \mathbb{P}(\mathbb{R}^n)
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$\Gamma$ discrete $< \text{SL}(n, \mathbb{R})$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$ if there exist:

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- $C \subset \Omega$ convex, $\Gamma$-invariant with $C/\Gamma$ compact $\neq \emptyset$

($\partial \Omega$ is $C^1$ and contains no segment)
Theorem (Danciger–Guéritaud–K. 2017)

Let $\Gamma$ be a discrete subgroup of $\text{SL}(n, \mathbb{R})$ preserving a properly convex open subset $\Omega$ of $\mathbb{P}(\mathbb{R}^n)$. Then

$$\Gamma \subset \text{SL}(n, \mathbb{R}) \text{ is } P_1-\text{Anosov} \iff \Gamma \text{ is strongly convex cocompact in } \mathbb{P}(\mathbb{R}^n)$$

$p$-Anosov $\Gamma$ do not preserve any convex set in $\mathbb{P}(\mathbb{R}^n)$.

Trick: pass to $\mathbb{P}(\mathbb{R}^n (n+1)/2) \cong \mathbb{P}(\{\text{quad. forms on } \mathbb{R}^n\})$. 

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Application (Danciger–Guéritaud–K., Zimmer)

If $n$ odd, then for any Hitchin $\rho : \pi_1(S) \to \text{SL}(n, \mathbb{R})$, $\rho(\pi_1(S))$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$. 

$\Omega$ “sufficiently regular” $\text{Im}(\xi_1)$ $\mathbb{C}$ convex compact mod $\Gamma$
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\[P \Rightarrow \text{SL}(n, R) \text{ is } P_1\text{-Anosov} \iff \Gamma \text{ is strongly convex cocompact in } \mathbb{P}(\mathbb{R}^n).\]
Section 5

Beyond Anosov subgroups
Projective convex cocompactness for nonhyperbolic groups

Definition \( \Gamma \) is strongly convex cocompact in \( \mathbb{P}(\mathbb{R}^n) \) if there exist:

- \( \Omega \) properly convex, \( \Gamma \)-invariant open \( \subset \mathbb{P}(\mathbb{R}^n) \) "sufficiently regular",
- \( C \subset \Omega \) convex, \( \Gamma \)-invariant with \( C/\Gamma \) compact \( \neq \emptyset \) with \( C \) large enough (\( C \) contains all accumulation points of \( \Gamma \)-orbits of \( \Omega \)).

\( \Omega \cap C \) convex compact mod \( \Gamma \)

Theorem (Danciger–Guéritaud–K. 2017)

- Convex cocompactness in \( \mathbb{P}(\mathbb{R}^n) \) is an open condition
- Good behavior under duality, embedding into larger \( \mathbb{P}(\mathbb{R}^N) \), ...
- Convex cocompact and Gromov hyperbolic \( \iff \) strongly convex cocompact
Γ discrete $< \text{SL}(n, \mathbb{R})$ is strongly convex cocompact in $\mathbb{P}(\mathbb{R}^n)$ if there exist:

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Definition

Γ discrete < SL(n, R) is strongly convex cocompact in \( \mathbb{P}(\mathbb{R}^n) \) if there exist:

- Ω properly convex, Γ-invariant open \( \subset \mathbb{P}(\mathbb{R}^n) \) “sufficiently regular”, and
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Theorem (Danciger–Guéritaud–K. 2017)

Convex cocompactness in \( \mathbb{P}(\mathbb{R}^n) \) is an open condition

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\(\overline{\mathcal{C}}\) contains all accumulation points of Γ-orbits of Ω

Theorem (Danciger–Guéritaud–K. 2017)

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Projective convex cocompactness for nonhyperbolic groups

**Definition**

\[ \Gamma \text{ discrete } < \text{SL}(n, \mathbb{R}) \text{ is strongly convex cocompact in } \mathbb{P}(\mathbb{R}^n) \text{ if there exist:} \]

1. \( \Omega \) properly convex, \( \Gamma \)-invariant open \( \subset \mathbb{P}(\mathbb{R}^n) \) “sufficiently regular”, and
2. \( C \subset \Omega \) convex, \( \Gamma \)-invariant with \( C/\Gamma \) compact \( \neq \emptyset \) with \( C \) large enough

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(\overline{C} \text{ contains all accumulation points of } \Gamma\text{-orbits of } \Omega)
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**Theorem (Danciger–Guéritaud–K. 2017)**

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Example: convex cocompact reflection groups
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Coxeter group: $W = \langle s_1, \ldots, s_N \mid (s_is_j)^{m_{i,j}} = 1 \rangle$,
where $m_{i,i} = 1$ and $m_{i,j} \in \{2, 3, 4, \ldots \} \cup \{\infty\}$ for $i \neq j$
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\( W \) Gromov hyperbolic (Moussong) \( \iff \) \( W \) does not contain a subgroup \( \simeq \mathbb{Z}^2 \)

\( W \) hyp. \( \iff \) \( W \) not hyp.

(Benoist)
Example: convex cocompact reflection groups

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Theorem (Danciger-Guéritaud-K., Lee-Marquis, 2016-2018)

Assume \( W \) infinite (irreducible).

- If \( W \) is Gromov hyperbolic, then \( \exists \rho : W \to \text{SL}^{\pm}(N, \mathbb{R}) \) injective with \( \rho(W) \) strongly convex cocompact reflection group.
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- If $W$ is Gromov hyperbolic, then $\exists \rho : W \rightarrow \text{SL}^\pm(N, \mathbb{R})$ injective with $\rho(W)$ strongly convex cocompact reflection group. $\rightarrow P_1$-Anosov subgroup
Example: convex cocompact reflection groups

Coxeter group: \( W = \langle s_1, \ldots, s_N \mid (s_is_j)^{m_{i,j}} = 1 \rangle, \) where \( m_{i,i} = 1 \) and \( m_{i,j} \in \{2, 3, 4, \ldots\} \cup \{\infty\} \) for \( i \neq j \)

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\[ \text{hyp.} \quad \xymatrix{ 4 \ar@{-}[rr] & & 4 \ar@{-}[rr] & & 4 \ar@{-}[rr] & & \tilde{A}_3 } \]

\[ \text{not hyp.} \quad \xymatrix{ 3 \ar@{-}[rr] & & 3 \ar@{-}[rr] & & 3 \ar@{-}[rr] & & \tilde{A}_5 } \]


Assume \( W \) infinite (irreducible).

- If \( W \) is Gromov hyperbolic, then \( \exists \rho : W \to \text{SL}^\pm(N, \mathbb{R}) \) injective with \( \rho(W) \) strongly convex cocompact reflection group. \( \rightarrow \) \( P_1 \)-Anosov subgroup

- In general, \( \exists \rho : W \to \text{SL}^\pm(n, \mathbb{R}) \) injective with \( \rho(W) \) convex cocompact reflection group for some \( n \leftrightarrow \) any subgroup \( \simeq \mathbb{Z}^2 \) of \( W \) is contained in a standard subgroup \( W_I = \langle s_i \rangle_{i \in I} \) of type \( \tilde{A}_k \).
Conclusion

Discrete subgroups of $G = \text{SL}(n, \mathbb{R})$ ▶ lattices and their subgroups ▶ Anosov subgroups ▶ good behavior: deformations, dynamical properties, important role in higher Teichmüller theory, many examples, ...

▶ geometric interpretation: $P^1$-Anosov $\iff$ strongly convex cocompact in $\text{P}(\mathbb{R}^n(\mathbb{R} + 1)^2) ▶ ...

One approach: relax strong convex cocompactness: ▶ allowing for cusps (Crampon–Marquis, Kapovich–Leeb, ...) ▶ convex cocompactness (not strong) (Danciger–Guéritaud–K.)

Questions: Dynamical properties? Other classes of discrete subgroups?
Conclusion

Discrete subgroups of $G = \text{SL}(n, \mathbb{R})$
Conclusion

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- lattices

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