Homology cobordism and triangulations

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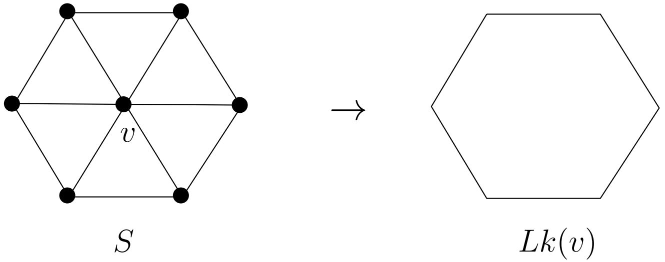
Question (Kneser 1924): *Does every topological manifold admit a triangulation?*

We can ask this about arbitrary triangulations, or about the more natural PL (combinatorial) triangulations.

Triangulations of manifolds

A triangulation is called *combinatorial (or PL)* if the link of every vertex is (PL homeomorphic to) a sphere. Clearly, every space that admits a combinatorial triangulation is a manifold.



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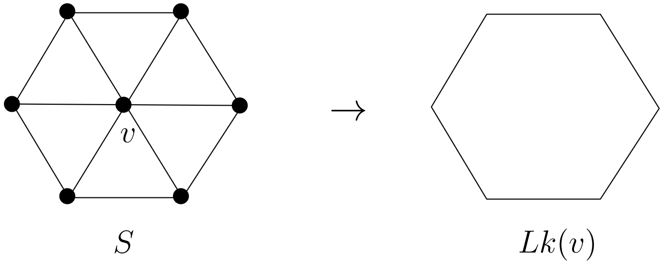
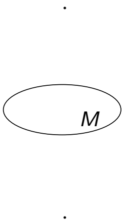


PL triangulations are equivalent to PL structures on the manifold (up to isomorphism).

Triangulations of manifolds

However, manifolds can have non-PL triangulations. **Example:** the double suspension of a homology sphere M^n with $\pi_1(M) \neq 1$.



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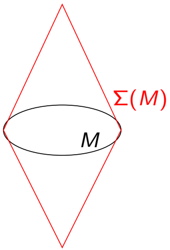

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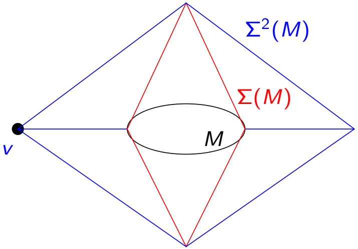

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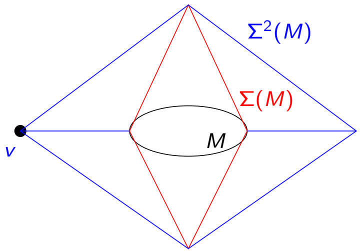

We have $\Sigma^2 M \cong S^{n+2}$ (cf. **Edwards, Cannon** $\sim$ 1970s), but $Lk(v) = \Sigma M$ is not a manifold.

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dim ≤ 3: unique smooth and PL structures

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- There exist non-PL manifolds
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- If they exist, PL structures are classified by elements in $H^3(M; \mathbb{Z}/2)$ (failure of the Hauptvermutung for manifolds)

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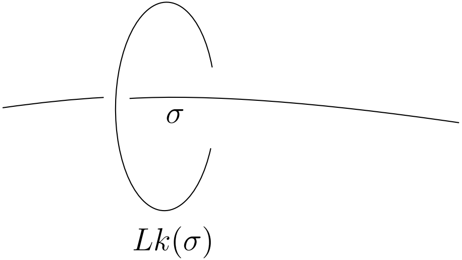
dim 4 (cf. **Donaldson, Freedman** $\sim$ 1982): smooth (=PL) structures may not exist, or may not be unique

Arbitrary triangulations

dim 4: (**Casson**) the Freedman E_8 manifold is non-triangulable dim 4: (**Casson**) the Freedman E_8 manifold is non-triangulable

dim ≥ 5: The idea is to look at the possible links of simplices of codimension $n+1$. They are n-dimensional homology spheres.

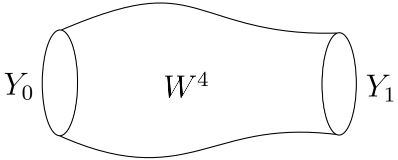


We can rephrase triangulation questions in terms of the n-dimensional *homology cobordism group*.

The homology cobordism group

$$\Theta^n_{\mathbb{Z}} = \{Y^n \text{ oriented, PL}, H_*(Y) = H_*(S^n)\} \;/\; \sim,$$

$$Y_0 \sim Y_1 \iff \exists \text{ compact, oriented, PL } W^{n+1} \text{ with } \partial W = (-Y_0) \cup Y_1 \text{ and } H_*(W, Y_i; \mathbb{Z}) = 0.$$



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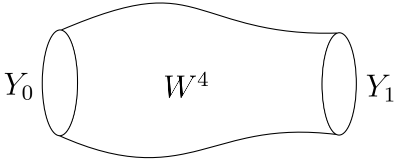

It turns out that $\Theta^n_{\mathbb{Z}} = 0$ for $n \neq 3$, but $\Theta^3_{\mathbb{Z}} \neq 0$.

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Rokhlin homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$, $\mu(Y) = \sigma(W)/8 \pmod 2$ where W is any compact, smooth, spin 4-manifold with boundary Y.

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Unknown: Does it have torsion? A $\mathbb{Z}^\infty$ summand? Is it $\mathbb{Z}^\infty$?

Triangulations of manifolds in dim ≥ 5

By the work of **Galewski-Stern** and **Matumoto** in the 1970s:

- There exist non-triangulable manifolds in dim $\geq 5 \iff$ the exact sequence

$$0 \longrightarrow \ker(\mu) \longrightarrow \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

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- Triangulations (if they exist) are classified by elements in $H^4(M; \ker(\mu))$.

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2. Seiberg-Witten theory
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In this talk we will discuss (2) and (3).

Seiberg-Witten equations

- system of nonlinear PDEs, elliptic (mod gauge), sensitive to the smooth structure (in dim.4)

- in the presence of a spin structure, they have a Pin(2) symmetry, where

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- in dim 3, we use them to obtain Seiberg-Witten Floer homology (cf. **Kronheimer-Mrowka, Frøyshov, Marcolli-Wang, M.**) and in fact a Seiberg-Witten Floer stable homotopy type (**M.**)

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- In retrospect, all the homology cobordism invariants from SW theory can be obtained through the *local equivalence group* (cf. **Stoffregen**)

The local equivalence group

There exist homomorphisms of Abelian groups

$$\Theta^3_{\mathbb{Z}} \to \mathcal{LE} \to \mathcal{CLE}$$

$$[Y] \to [\mathrm{SWF}(Y)] \to [C_*(\mathrm{SWF}(Y); \mathbb{F})], \quad \mathbb{F} = \mathbb{Z}/2$$

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$\mathcal{CLE}$ (chain local equivalence) is defined similarly, with chain complexes.

Numerical invariants

Prototype: The Frøyshov homomorphism

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obtained from the S^1-equivariant Seiberg-Witten Floer (co)homology of Y, which is a module over

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For the Poincare sphere we have $\delta(P) = 1$. The existence of a surjective homomorphism $\delta : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ shows that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}$ summand:

$$\Theta^3_{\mathbb{Z}} = [P] \oplus \ker(\delta).$$

Pin(2) invariants

Similarly to the Frøyshov invariant, we can use $H^*_{\mathrm{Pin}(2)}(\mathrm{SWF}(Y);\mathbb{F})$ to construct maps (not homomorphisms)

$$\Theta^3_{\mathbb{Z}} \longrightarrow \mathcal{LE} \longrightarrow \mathcal{CLE} \overset{\alpha,\beta,\gamma}{\dashrightarrow} \mathbb{Z},$$

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Indeed: $2[Y] = 0 \Rightarrow Y \sim -Y \Rightarrow \beta(Y) = \beta(-Y) = -\beta(Y) \Rightarrow \beta(Y) = 0 \Rightarrow \mu(Y) = 0.$

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Alternate construction of α, β, γ: **F. Lin**, 2014.

More numerical invariants

$$\begin{array}{ccccccc} \Theta^3_{\mathbb{Z}} & \longrightarrow & \mathcal{LE} & \longrightarrow & \mathcal{CLE} & \xrightarrow{\delta} & \mathbb{Z}, \\ & & \downarrow{\scriptstyle \kappa_i,\kappa o_i} & & \downarrow{\scriptstyle \bar{\delta},\underline{\delta}} & \searrow{\scriptstyle \alpha,\beta,\gamma} & \\ & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

where $\bar{\delta}, \underline{\delta}$ come from $\mathbb{Z}/4$-equivariant SWFH:

$$\mathbb{Z}/4 = \{1, -1, j, -j\} \subset \mathrm{Pin}(2) = \mathbb{C} \oplus j\mathbb{C}$$

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$$\kappa_i, i = 0, 1; \quad \kappa o_i, i = 0, \dots, 7$$

come from Pin(2)-equivariant K-theory and KO-theory (**M., J. Lin**)

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Open problem: Describe the structure of $\mathcal{LE}, \mathcal{CLE}$ in general, and use it to understand more about $\Theta^3_{\mathbb{Z}}$.

Heegaard Floer homology

Ozsváth-Szabó, 2001: $Y^3 \rightsquigarrow \widehat{HF}(Y), HF^+(Y), HF^-(Y)$ using Lagrangian Floer homology on the symmetric product of a Heegaard surface for Y

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- replacement for SW theory:

$$HF^+ = \widecheck{HM} = H_*^{S^1}(\text{SWF}), \quad \widehat{HF} = \widetilde{HM} = H_*(\text{SWF})$$

cf. (**Kutluhan-Lee-Taubes, Taubes + Colin-Ghiggini-Honda**) + **Lidman-M.**

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One can recover the Frøyshov invariant δ using HF^+ (**Ozsváth-Szabó**'s d-correction term).

Involutive Heegaard Floer homology

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It suffices to give invariants $\bar{\delta}, \underline{\delta} : \Theta^3_{\mathbb{Z}} \dashrightarrow \mathbb{Z}$, computable for Seifert fibrations, large surgeries on alternating knots, connected sums of these (**Hendricks-M.-Zemke, Dai-M., Dai-Stoffregen** 2016-7).

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- new proofs that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ subgroup (**Stoffregen** using Pin(2)-equiv. SW theory; **Dai-M.** using *HFI*)

An analogue of $\mathcal{CLE}$

There is a homomorphism

$$\Theta^3_{\mathbb{Z}} \to \mathfrak{I}, \quad [Y] \to [CF^-(Y), \text{ the conjugation involution}]$$

where $\mathfrak{I} = \{$free $\mathbb{F}[U]$-complexes C_* with automorphism $\iota, \ \iota^2 \simeq \text{id}$, $U^{-1}H_*(C) \cong \mathbb{F}[U, U^{-1}]\}/\sim$

$C_* \sim D_* \iff \exists f : C \to D, \ g : D \to C$ module homomorphisms that induce $\cong$ on $U^{-1}H_*$, and such that $f\iota \simeq \iota f, \ g\iota \simeq \iota g$.

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Open question: *What is $\mathfrak{I}$ as an Abelian group? Can we use it to study $\Theta^3_{\mathbb{Z}}$, e.g. to show it has a $\mathbb{Z}^\infty$ summand?*