Cannon-Thurston Maps

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Geometric Structures on Surfaces

- **Differential Geometry:** Constant curvature metrics: $+1$ ($g = 0$), $0$ ($g = 1$), $-1$ ($g \geq 2$).
- **Lie groups:** Discrete faithful representation 
  \[ \rho : \pi_1(S) \to PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm I\} = Isom^+(H^2). \]
- **Complex Geometry:** Riemann surfaces: transition functions complex analytic.
- **Algebraic Geometry:** Solution sets to algebraic equations: (Complex) 1 dimensional smooth varieties in $\mathbb{C}P^n$.

Poincaré-Koebe-Klein uniformization theorem establishes a dictionary between these structures.
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Discrete faithful representation of Coxeter group.

Constant curvature -1 metric on closed surface $S$ of genus at least 2.
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PSL(2, C) = Isom^+(H^3)

Look at space of discrete faithful \( \rho : \pi_1(S) \to PSL(2, \mathbb{C}) \) equipped with the usual (algebraic) topology of (pointwise) convergence. Let \( \Gamma = \rho(\pi_1(S)) \) – Kleinian surface group.

**Definition (Algebraic topology on space of representations)**

A sequence of representations \( \rho_n : \pi_1(S) \to PSL_2(\mathbb{C}) \) is said to converge **algebraically** to \( \rho_\infty : \pi_1(S) \to PSL_2(\mathbb{C}) \) if for all \( g \in \pi_1(S) \), \( \rho_n(g) \to \rho_\infty(g) \) in \( PSL_2(\mathbb{C}) \).

The collection of conjugacy classes of discrete faithful representations of \( \pi_1(S) \) into \( PSL_2(\mathbb{C}) \) equipped with the algebraic topology is denoted as \( AH(S) \).

In dimension 2 (for \( PSL_2(\mathbb{R}) \)), this is Teichmüller space.
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Hyperbolic structures on $S \times \mathbb{R}$: Geometry

Theorem (Topological tameness) (Thurston-Bonahon): For $\Gamma = \rho(\pi_1(S))$ - a Kleinian surface group, $M = \mathbb{H}^3/\Gamma$ is homeomorphic to a product $S \times \mathbb{R}$.

But geometrically, a lot of variety.
So 3-dimensional analog of Teichmüller theory becomes the study of hyperbolic structures on $M = S \times \mathbb{R}$ up to isometry.
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So 3-dimensional analog of Teichmüller theory becomes the study of hyperbolic structures on $M = S \times \mathbb{R}$ up to isometry.
Let \( i : S \to M \) be a homotopy equivalence (embedding),
\( o \in \mathbb{H}^2 = \tilde{S} \) be a base-point,
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Dichotomy of representations

Broadly, 3d hyperbolic geometric structures on $S \times \mathbb{R}$ are of two kinds.

- Quasi-Fuchsian/ Convex cocompact/ undistorted/ quasi-isometrically (qi) embedded:
  Distances in $H^2$ (denote $d_2$) and $H^3$ (denote $d_3$) are linearly comparable: There exist $(k, \epsilon)$ such that

$$\frac{1}{k} d_2(g.o, h.o) - \epsilon \leq d_3(\rho(g).O, \rho(h).O) \leq k d_2(g.o, h.o) + \epsilon.$$

$QF(S) = Teich(S) \times Teich(S)$ (Bers’ simultaneous uniformization theorem, 1960) Diagonal=Fuchsian

- Limits of the above in the algebraic topology. (Bers’ density conjecture proved by Brock-Canary-Minsky, 2012) These are not qi-embedded.

Accordingly, ends of $H^3/\Gamma = S \times \mathbb{R} = M$ can also be quasi-Fuchsian or degenerate.
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Limit set $\Lambda_{\Gamma} =$ Set of accumulation points in $\hat{\mathbb{C}}$ of $\Gamma \cdot O$ for some (any) $O \in \mathbb{H}^3$: the locus of chaotic dynamics of the $\Gamma$--action on $S^2$.

For a quasi-Fuchsian group (subgroup of $PSL_2(\mathbb{R})$), limit set is a quasi-circle.
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**Definition (From complex analytic to geometric perspective:)**

The convex hull $CH_\Gamma$ of $\Lambda_\Gamma$ is the smallest non-empty closed convex subset of $H^3$ invariant under $\Gamma$.

Let $M = H^3/\Gamma$. The quotient of $CH_\Gamma$ by $\Gamma$ is called the convex core $CC(M)$ of $M$.

For quasi-Fuchsian groups, $CC(M)$ is homeomorphic to $S \times [-1, 1]$.

**Limits of quasi-Fuchsian groups:**
Thickess of Convex core $CC(M)$ tends to infinity.
2 possibilities: Degenerate only $\tau_1$. Degenerate both $\tau_1, \tau_2$.
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*The convex hull* $CH_\Gamma$ of $\Lambda_\Gamma$ is the smallest non-empty closed convex subset of $H^3$ invariant under $\Gamma$.

Let $M = H^3/\Gamma$. The quotient of $CH_\Gamma$ by $\Gamma$ is called the convex core $CC(M)$ of $M$.

For quasi-Fuchsian groups, $CC(M)$ is homeomorphic to $S \times [-1, 1]$.

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For all representations in $\text{AH}(S)$, $\tilde{i} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ extends to a continuous map $\partial i : S^1 \rightarrow S^2$

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Cannon-Thurston maps identify precisely the end-points of ending laminations. 

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Ending lamination determines $\rho$ up to conjugacy.

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Connected limit sets of finitely generated discrete subgroups of $\text{PSL}_2(\mathbb{C})$ are locally connected.

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Let \( H \subset G \) be a hyperbolic subgroup of a hyperbolic group. Does the inclusion \( i : \Gamma_H \to \Gamma_G \) of Cayley graphs extend to the boundary, giving \( \partial i : \partial H \to \partial G \)?

No, in this generality (Baker-Riley 2013)
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Surface group representations in other non-compact semisimple Lie groups $G$: Anosov representations (Labourie) give $\pi_1(S)$–equivariant embeddings from $S^1$ to $G/P$. Analog of $QF(S)$. Work of Kapovich-Leeb-Porti; Gueritaud-Guichard-Kassel-Wienhard.

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