Stability conditions in symplectic topology

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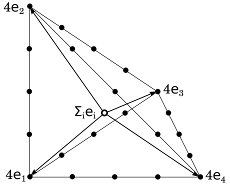
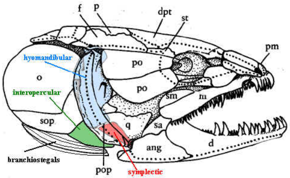


Oxford English Dictionary – definition of symplectic

adj. *Etym.* from Greek $\sigma\upsilon\mu\pi\lambda\epsilon\kappa\tau\iota\kappa o\varsigma$. Twining or plaiting together, copulative.



Skull of Amia showing neopterygian features. Dermal bones of the cheek and lower jaw have been shifted right to expose the quadrate. The symplectic lies largely behind the quadrate ("q"), and the hyomandibular behind the preopercular ("pop").

1. Anatomy, Zoology Epithet of a bone in the suspensorium in the skull of fishes (image ©palaeos.com)

A symplectic structure on X is $\omega \in \Omega^2(X)$ with $d\omega = 0$, pointwise non-degenerate skew-form on $T_x X$ ($\Rightarrow \dim_{\mathbb{R}}(X)$ even). *No local invariants*: locally $(\mathbb{R}^{2n}, \sum dx_j \wedge dy_j)$.

Key examples: oriented surface Σ, smooth algebraic variety X.
Lagrangian submanifolds: $L^n \subset X^{2n}, \omega|_L \equiv 0$; circles in Σ, $X(\mathbb{R})$.

Where are we going?

Let (X, ω) be a symplectic manifold and

- $G(X, \omega) = \pi_0 \mathrm{Symp}_{ct}(X, \omega)$, the *symplectic mapping class group*.
- $I(X, \omega) = \ker\, (G(X, \omega) \to \mathrm{Aut}\, H^*(X; \mathbb{Z}))$, the *symplectic Torelli group*

Theorem [Sheridan, S.] For $k \geqslant 2$, there is a closed simply-connected symplectic $2k$-manifold (X, ω) for which $I(X, \omega)$ is infinitely generated.

Broad aim: understand the countable groups $G(X, \omega)$, $I(X, \omega)$.

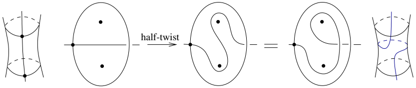
— Finitely presented? Arithmetic? Hyperbolic?

— Any constraints? Could any countable group occur?

The success story: the classical mapping class group

$$\Gamma_g = \underbrace{\pi_0 \mathrm{Diff}^+(\Sigma_g)}_{\text{topology}} = \underbrace{\mathrm{Out}(\pi_1(\Sigma_g))}_{\text{group theory}} = \underbrace{\pi_1(\text{Moduli space of curves})}_{\text{algebraic geometry}}$$

- *algebraic* – finitely presented; $I(\Sigma_g)$ infinitely generated if $g = 2$; generated by Dehn twists or torsion elements, residually finite...
- *geometric* – δ-hyperbolic, not arithmetic, rigidity of $\Gamma_g \to \Gamma_h$, quasimorphisms...



NB Dehn twist in annulus T^*S^1 generalises to twist $\tau_{S^n} \in G(T^*S^n, \omega)$.

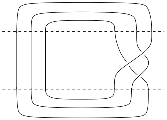
First symplectic information: monodromy of families

— Given algebraic X, moving in a moduli space $\mathcal{M}_X$, symplectic parallel transport identifies fibres of the universal family $\rightsquigarrow$ $\pi_1 \mathcal{M}_X \to G(X, \omega)$.

Q. Is this an isomorphism? (Or: can we characterise the image?)

Example: p a degree m polynomial, distinct roots: $X = \{x^2 + y^2 + p(z) = 0\} \subset \mathbb{C}^3$ is smooth, moves in family over $\mathcal{M}_X = \mathrm{Conf}_m(\mathbb{C})$.

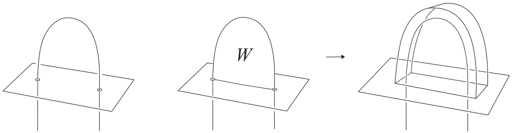
$$\begin{array}{ccc} Br_m & \xrightarrow[\rho]{\simeq} & G(X,\omega) \\ \downarrow & & \downarrow \\ \mathrm{Sym}_m & \longrightarrow & \pi_0 \mathrm{Diff}_{ct}(X) \end{array}$$



"Classical": show ρ is 1-1. Khovanov-Seidel '02. *Harder*: find tools to show ρ *onto*. Wu '13

The smooth story: $\Gamma(X) = \pi_0 \mathrm{Diff}(X)$, $n \geqslant 5$

- $\Gamma(S^n)$ is finite group of exotic $(n+1)$-spheres; (Kervaire-Milnor '63)
- $\Gamma(T^n) = (\mathbb{Z}/2)^\infty \oplus (\mathrm{rest})$ (Hatcher & Hsiang '77).
- $\pi_1(X) = 0 \Rightarrow \Gamma(X)$, $\ker(\Gamma(X) \to Aut(H^*(X)))$ commensurable with arithmetic groups: finitely presented, no divisible subgroups... so $\Gamma(X) \not\supset \mathbb{Q}$ (Sullivan '77).



Understand $\Gamma(X)$ via action on "minimal model", a dga quasi-iso^c to $C^*(X;\mathbb{Q})$, so acts via $\mathrm{HtEq}(X)$. Whitney trick cancels intersections between submanifolds, showing $\{\text{smooth info}\} \to \{\text{homotopy info}\}$ finite-to-one when $\pi_1(X) = 0$. (image © Conant *et al*)

Imitating the two-dimensional case

Γ_g acts on Teichmüller space $\mathcal{T}_g$ of curvature -1 metrics on Σ, or complex of curves $\mathcal{C}_g$: vertices $\Leftrightarrow$ simple closed curves, edge if curves can be disjoined.

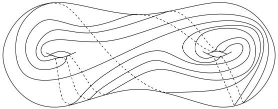


Figure 1 Thurston's "typical curve".

Geometric intersection numbers, refines H^*.

Key insight: $\mathcal{T}_g$ contractible, $\mathcal{C}_g$ δ-hyperbolic.

Given (X, ω) symplectic, can form a graph $S(X)$:

- Vertices $\Leftrightarrow$ Lagrangian spheres up to Lagrangian isotopy
- Edges $\Leftrightarrow$ spheres are disjoinable by Lagrangian isotopy

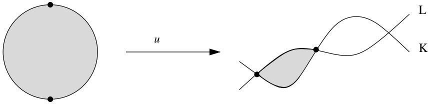
$G(X, \omega)$ acts by simplicial automorphisms... but $S(X)$ too mysterious to be useful.

For $X_d \subset \mathbb{P}^3$ degree d, $S(X_1) = \emptyset$, $S(X_2) = \{\text{point}\}$ (Hind 2004), $S(X_d) = (??)$ for $d \geqslant 3$

Floer theory of (oriented) Lagrangian submanifolds

For $L, K \rightsquigarrow$ Floer cohomology $HF^*(L,K)$, counting solutions $\overline{\partial}_J(u) = 0$ to non-linear PDE. *Lagrangian-isotopy-invariant refinement of H_*-intersection.*

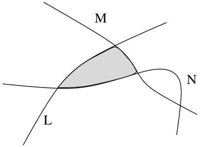
- $CF^*(L,K) = k\langle L \pitchfork K \rangle$
- $\langle \mu^1(p), q \rangle = \#\mathcal{M}(p,q)$



— "Whitney trick only for holomorphic discs"

— $\chi(HF) = [L]\cdot[K]$, spectral sequence $H^*(L;k) \Rightarrow HF^*(L,L)$

Product $HF(M,N) \otimes HF(L,M) \xrightarrow{\mu^2} HF(L,N)$ counting holomorphic triangles; higher products $\{\mu^{k-1}\}$ count holomorphic k-gons

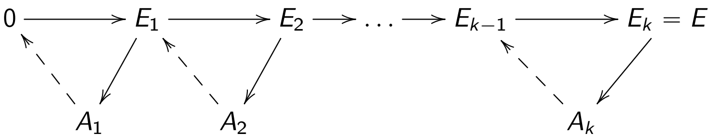


Fukaya category $\mathcal{F}(X,\omega)$, a (triangulated A_∞-)category with Lagrangians as objects.

Bridgeland's stability conditions

Triangulated $\mathcal{D}$ has a complex manifold ("Teichmüller space") of pairs $\sigma = (Z, \{P\})$:

- $Z : K(\mathcal{D}) \to \mathbb{C}$; $K(\mathcal{D}) = \mathbb{Z}\,\mathrm{Ob}(\mathcal{D})/\langle\text{triangles}\rangle$ (target is $\mathbb{C}$ even if $\mathcal{D}$ linear over k);
- subcategories $P(\phi)_{\phi\in\mathbb{R}}$, with $Z(E) \in \mathbb{R}_{>0}e^{i\phi}$ if $E \in P(\phi)$, s.t.



exists for $0 \neq E \in \mathcal{D}$, with $A_j \in P(\phi_j)$, $\phi_1 > \phi_2 > \cdots > \phi_k$.

Motivation: (X, J, ω) Calabi-Yau. Choose holomorphic n-form Ω. Set $Z(L) = \int_L \Omega$. $P(\phi) = \{$special Lagrangians, $\Omega|_L = e^{i\phi} \cdot \mathrm{vol}_L\}$. Mean curvature flow $\{L_t\}$ has $d\phi(L_t)/dt = \Delta(\phi(L_t)) \overset{?}{\rightsquigarrow}$ sLags A_j, constant ϕ_j *(finite-time singularities! impossibly hard!)*

Conjecturally $\widetilde{\mathcal{M}}_X(J, \Omega) \hookrightarrow \mathrm{Stab}(\mathcal{F}(X, \omega))$

Contractibility?

Suppose $K(\mathcal{D}) \cong \mathbb{Z}^d$ is finite rank; $\mathrm{Stab}(\mathcal{D})$ is a d-dimensional complex manifold, locally $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$. $\mathrm{Auteq}(\mathcal{D})$ *acts discretely with finite (non-generic) stabiliser*.

Conjecture: $\mathrm{Stab}(\mathcal{D})/\mathrm{Auteq}(\mathcal{D})$ has (orbifold) $K(\pi, 1)$ components.

Some finitely presented groups can't act effectively on a finite dimensional contractible manifold, if say virtual cohomological dimension is ∞ (e.g. Thompson's group F).

New strategy: Show $\mathrm{Stab}(\mathcal{F}(X))$ is 1-connected, relate $\mathrm{Stab}/\mathrm{Auteq}$ to $\mathcal{M}_X$. Hope

$$\pi_1(\mathcal{M}_X) \to \mathrm{Auteq}(\mathcal{F}(X)) \to \mathrm{Deck}(\mathrm{Stab}(\mathcal{F}(X)) = \pi_1(\mathcal{M}_X)$$

shows $G(X, \omega) = \pi_1(\mathcal{M}_X) \rtimes N$: monodromy of algebraic families splits off the symplectic mapping class group, so *intrinsic*; "negligible" N acts trivially on $\mathcal{F}(X)$.

K3 surfaces & mirror symmetry

Theorem (Bayer, Bridgeland) If Y is a Picard rank one complex K3 surface, $\mathrm{Stab}(\mathcal{D}(Y))$ has Auteq-invariant contractible component.

Say (X, ω) is homologically mirror to Y if $\mathcal{F}(X, \omega) \simeq \mathcal{D}(Y)$. Naively Y is a "moduli space of objects (tori) in $\mathcal{F}(X)$", e.g. $\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong H^*(T^2)$.

[Actual proof of HMS is more *ad hoc* (Seidel, Sheridan). Delicacy: Y usually defined over $\Lambda = \mathbb{C}((q^{\mathbb{R}}))$.]

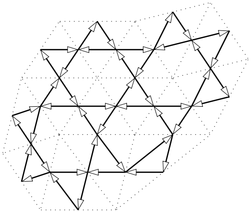
B-B $\Rightarrow \mathrm{Stab}(\mathcal{F}(X))/\mathrm{Auteq} \simeq \mathcal{M}_X(J, \Omega)$ is $\mathbb{C}^*$- bundle over a modular curve; covering for *Aut* $HH^*(\mathcal{F})$ is infinitely punctured upper-half-plane, & $I(X, \omega) \twoheadrightarrow \pi_1(\mathfrak{h} - \Delta)$.

Key argument in B-B: use a flow on $\mathrm{Stab}(\mathcal{D}(Y))$ decreasing $\phi_{max}(\mathcal{O}_p) - \phi_{min}(\mathcal{O}_p)$ to constant, contracting $\mathrm{Stab}(\mathcal{D}(Y))$ to locus where all $\mathcal{O}_p$ are stable of same phase.

Moral: *don't fix (J, Ω) and flow L, fix L and flow (J, Ω) instead??*

Classical Teichmüller theory revisited?

Ginzburg: a quiver Q (directed graph) with potential W (formal combination of cycles) canonically defines a triangulated A_∞-category $\mathcal{C}(Q, W)$.



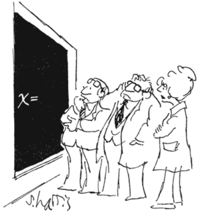
Triangulation of Σ_g with $d \geqslant 2$ punctures $\rightsquigarrow$ quiver, potential $W = \sum(\circlearrowleft \text{triangles}) - \sum(\circlearrowleft \text{polygons})$ $\rightsquigarrow$ category $\mathcal{C}(g, d)$. Fact: There is a Calabi-Yau 3-fold $X \to \Sigma_g$ with $\mathcal{C}(g, d) \hookrightarrow \mathcal{F}(X, \omega)$

Theorem [Bridgeland-S.] $\mathrm{Stab}(\mathcal{C}(g, d))/\mathrm{Auteq} \cong \mathrm{Quad}(g, d)$ (meromorphic quadratic differentials with d poles order $\leqslant 2$ and simple zeroes).

Stable objects $\Leftrightarrow$ saddle connections of associated flat surface $\Leftrightarrow$ sLag S^3's in X.

Cartoon outlook

Pessimistic: Theory limited.
$\mathrm{Stab}(\mathcal{F}(X))$ only defined if $2c_1(X) = 0$.
10 years to prove $\mathrm{Stab}(\mathcal{D}(CY3)) \neq \emptyset$.
Contractibility still out of reach.



Optimistic: Quiver examples tractable.
New phenomena: infinite-generation.
HMS ideas separately compelling.
Lots of new things to try!

Thank you to my teachers, collaborators & the audience.