Flexibility in symplectic and contact geometry

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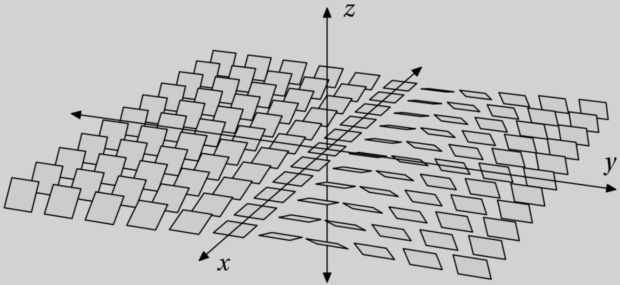
- ξ on Y determines and is determined by the structure ω on X, at infinity.

Example contact structure



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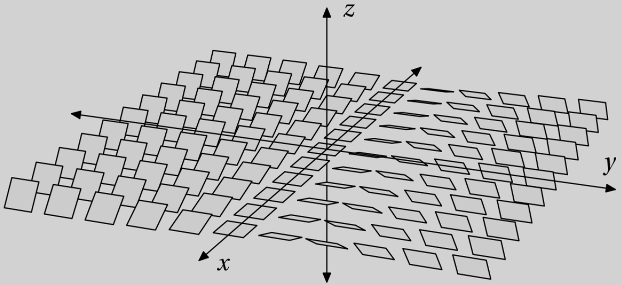

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Every contact manifold can be built by gluing copies of this model together via contactomorphisms.

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For a simple example, a *formal immersion* $M \to N$ consists of a bundle monomorphism $TM \to TN$.

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Symplectic and contact geometry one place where this distinction is especially interesting: often the geometry gets very close to the topology!

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Theorem (Casals–Pancholi–Presas 2015, indep. Etnyre)

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Let $\varphi_t : (D^{2n-3}, \xi_{\text{std}}) \to (D^{2n-3}, \xi_{\text{std}})$ be an isotopy through contactomorphisms. Because contact geometry is tied to Hamiltonian dynamics, every contact isotopy comes from an energy function, or Hamiltonian. That is, the Lie algebra of the contactomorphism group $\text{Cont}(D^{2n-3}_{\text{std}})$ is equal to $\mathfrak{cont}(D^{2n-3}_{\text{std}}) \cong C^\infty(D^{2n-3})$.

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Following from the elementary theory, by suspending any contact isotopy of D^{2n-3}_{std} we obtain a germ of a contact structure near S^{2n-2}. If the Hamiltonian generating this isotopy is positive everywhere, then the contact structure extends in an obvious way to B^{2n-1}.

Proof sketch, continued

By taking our original triangulation on Y to be sufficiently fine, we only need to consider contact structures on ∂B^{2n-1} which are of this suspension type, thus we need to understand how the Lie group $\mathrm{Cont}(D^{2n-3}_{\mathrm{std}})$ interacts with the positive cone on $\mathfrak{cont}(D^{2n-3}_{\mathrm{std}}) \cong C^\infty(D^{2n-3})$.

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Besides the adjoint action, we have an additional "monoid structure" coming from contactomorphism $D^{2n-3}_{\mathrm{std}} \# D^{2n-3}_{\mathrm{std}} \cong D^{2n-3}_{\mathrm{std}}$ (boundary connect sum). We can realize this monoid structure by doing ambient connect sums inside Y, in this way it becomes an additional tool to construct extensions.

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To complete the proof that any contact structure on ∂B^{2n-1} extends to the interior, we prove the non-existence of any "causality" on $\mathrm{Cont}(D^{2n-3}_{\mathrm{std}})$, which is compatible with the positive cone in $\mathfrak{cont}(D^{2n-3}_{\mathrm{std}})$, together with the adjoint and connect sum actions.

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Contact structures on compact manifolds have no moduli: any deformation of a contact structure is realized by an ambient isotopy. Thus, uniqueness is just parametric existence. That is, if ξ_0 and ξ_1 are contact structures which are homotopic among *formal* contact structures, and if our existence proof is smooth in families, we would obtain a family of contact structures ξ_t interpolating between ξ_0 and ξ_1, by which we conclude that ξ_0 is contactomorphic to ξ_1.

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Nevertheless, this approach is productive for a partial classification, those of overtwisted contact manifolds.

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This generalizes earlier work of [Eliashberg 1989], where the same result was established for 3–manifolds Y.

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Ultimately, the most essential property of overtwistedness is semi-locality: there is an overtwisted contact structure on the open ball, and any contact manifold containing an overtwisted open set is itself overtwisted.

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The general structure of Θ is poorly understood. For instance, we do not know if there are any units in this monoid, or zero divisors.

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If $\Lambda \subseteq Y = X \cap S^{2N-1}$ for $X \subseteq \mathbb{C}^N$, and $L = (-\varepsilon, \varepsilon) \times \Lambda \subseteq X$ is defined by the gradient flow of $|\cdot|^2|_X$, then L is Lagrangian if and only if Λ is Legendrian.

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Often the topology disagrees with the geometry: for instance there is a formal Lagrangian $S^3 \subseteq \mathbb{C}^3$, but there is no genuine Lagrangian in $\mathbb{C}^3$ diffeomorphic to S^3 [Gromov 1985].

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Similar to overtwistedness for contact structures, the most important property defining loose Legendrians is that it is semi–local.

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Let L be any closed 3*–manifold, then there is a Lagrangian embedding* $L \# S^2 \times S^1 \subseteq \mathbb{C}^3$.

These Lagrangians are additionally interesting because they are “exotic” from the perspective of Lagrangian Floer theory: there are no pseudo-holomorphic curves passing through a generic point.

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Theorem (Ekholm–Eliashberg–M.–Smith 2013, Ekholm–Smith 2016, Eliashberg–M. 2013)

Let $L \looparrowright \mathbb{C}^{2n}$ be an exact Lagrangian immersion, with only a single transverse double point. Then $\chi(L) = \pm 2$. If $\chi(L) = +2$, then L is diffeomorphic to S^{2n}. If $\chi(L) = -2$, then L can be diffeomorphic to any smooth manifold with the property that $TL \otimes \mathbb{C}$ is trivial.

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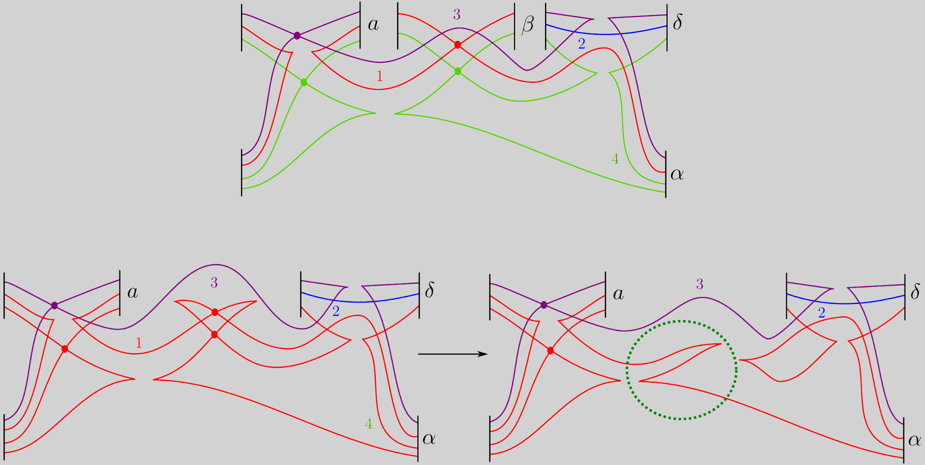

Trivializing a Koras–Russell cubic. (see [Casals–M.])