Acylindrically hyperbolic groups

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Studying groups as geometric objects

Two approaches to convert a group $G$ into a geometric object:

1. Fix a generating set $X$ of $G$ and study the Cayley graph $\Gamma = \text{Cay}(G, X)$ equipped with the combinatorial metric:

2. Let $S$ be a metric space and let $G \curvearrowright S$ (by isometries). Fix $s \in S$ and study the geometric structure of the $G$-orbit of $s$.

These approaches work especially well in the presence of certain "negative curvature" conditions.
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Definition (Gromov)

A geodesic metric space is **hyperbolic** if \( \exists \delta \geq 0 \) such that for any triangle with geodesic sides \( p, q, r \) and any \( x \in p \), we have \( d(x, q \cup r) \leq \delta \).
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A geodesic metric space is **hyperbolic** if there exists a constant $\delta \geq 0$ such that for any triangle with geodesic sides $p$, $q$, $r$ and any $x \in p$, we have $d(x, q \cup r) \leq \delta$.

**Examples.**

1. Any bounded space $S$ is hyperbolic with $\delta = \text{diam}(S)$.
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3. $\mathbb{H}^n$ is hyperbolic.
4. $\mathbb{R}^n$ is not hyperbolic for $n \geq 2$. 
A finitely generated group $G$ is *hyperbolic* if $\text{Cay}(G, X)$ is hyperbolic for some finite generating set $X$ of $G$. 

**Examples.**
1. Finite groups.
2. Free groups of finite rank.
3. $\pi_1(M)$ for any closed hyperbolic manifold $M$. 

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\begin{tikzpicture}
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\node [circle, fill=black, draw, line width=1.5mm, line cap=round, inner sep=10pt, fill opacity=0.5] (a) at (0,0) {hyperbolic groups};
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\[ \text{hyperbolic groups} \supset \text{relatively hyperbolic groups} \]

- $\text{MCG}(S_g)$
- $\text{Out}(F_n)$

\[ \vdots \]
Acylindrically hyperbolic groups

Definition (Bowditch)
An isometric action of $G$ on a metric space $S$ is acylindrical if

$\forall \varepsilon > 0 \exists R, N > 0 \forall x, y \in S \ d(x, y) \geq R \Rightarrow |\{g \in G \mid d(x, gx) \leq \varepsilon \land d(y, gy) \leq \varepsilon\}| \leq N$.

Examples:
$G \rtimes \text{pt}. \text{ Proper & cocompact } \Rightarrow \text{acylindrical}.$

$\text{MCG}(S_g) \rtimes \text{curve complex for } g \geq 2$ (Bowditch).
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$G$ is acylindrically hyperbolic if it admits a non-elementary acylindrical action on a hyperbolic space.

In our settings: non-elementary $\iff G$ is not virtually cyclic and has unbounded orbits.

Examples

Non-elementary hyperbolic and relatively hyperbolic groups.

$\text{Out}(F_n)$ (Bestvina-Feighn).

If $G = \pi_1(\text{closed irreducible 3-manifold})$, then $G$ is acylindrically hyperbolic, or is virtually solvable, or $M$ is Seifert fibered (Minasyan-Osin).

Groups of deficiency $\geq 2$ (Osin).
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1. **Group theoretic Dehn filling:** Generalization of Thurston’s theory of hyperbolic Dehn fillings in 3-manifolds.

2. **Small cancellation theory:** Proving embedding theorems and constructing groups with interesting properties.

3. **Measure theoretic rigidity:** Every a.h. group $G$ has plenty of quasi-cocycles $G \to \ell^2(G)$ (Hamenstadt, Hull-Osin, Bestvina-Bromberg-Fujiwara). It follows that $H^2_{b}(G, \ell^2(G)) \neq 0$ and Monod-Shalom rigidity theory for measure preserving actions applies.
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Dehn filling

$M$ is a compact 3-manifold, $\partial M = T^2$. Fix $s \in \pi_1(\partial M)$ and let $M(s) = M \cup \phi(S^1 \times D^2)$, where $\phi: \partial (S^1 \times D^2) \to \partial M$ is such that $\phi(\partial D^2) \in s$.

Theorem (Thurston) If $M \setminus \partial M$ admits a finite volume hyperbolic structure, then $M(s)$ is hyperbolic for all but finitely many $s$.

Group theoretic Dehn filling

$M$ a.h. group $G$

$\partial M$ hyperbolically embedded $H \leq G$

$s \in \pi_1(\partial M)$

$h \in H$

$M(s) G / \langle \langle h \rangle \rangle G$
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<table>
<thead>
<tr>
<th>$M$</th>
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</thead>
<tbody>
<tr>
<td>$\partial M$</td>
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**Examples.**

1. If $M$ is as in the Thurston theorem, then $\pi_1(\partial M) \hookrightarrow h \pi_1(M)$.
2. If $G = A * B$, then $A, B \hookrightarrow h G$.
3. If $G$ is finitely generated and a.h., then a random subgroup is virtually free and hyperbolically embedded (Maher-Sisto).
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Let $H \hookrightarrow_h G$. Then there exists finite $\mathcal{F} \subseteq H \setminus \{1\}$ such that for all $N \triangleleft H$ satisfying $N \cap \mathcal{F} = \emptyset$, we have

1. $\langle \langle N \rangle \rangle^G \cap H = N$; equivalently, the natural map $H/N \to G/\langle \langle N \rangle \rangle^G$ is injective.
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Remarks.

— Implies Thurston's theorem (modulo the geometrization conjecture).
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3. Structure of the kernel: Theorem (Dahmani-Guirardel-Osin, 2016) Under the assumptions of the main theorem, \( \langle \langle N \rangle \rangle \) is isomorphic to the free product of copies of \( N \). Implies solution of two open problems about mapping class groups.
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Constructing groups with interesting properties

Question (Burnside)

Is every finitely generated torsion group finite?

Answered negatively by Golod in 1964.

Naive idea

Let $F(x, y) = \{1, f_1, f_2, \ldots\}$ and $G = \langle x, y \mid f_n^1, f_n^2, \ldots \rangle$.

Clearly $G$ is torsion.

Why is it non-trivial???

Theorem (Gromov-Olshanskii)

If $G$ is non-elementary hyperbolic and $g \in G$, then $G/\langle\langle g^n \rangle\rangle$ is non-elementary for some $n >> 1$.

It follows that $G_k = \langle x, y \mid f_n^1, \ldots, f_n^k \rangle$ can be made non-elementary hyperbolic for all $k$. Therefore, $|G| = \infty$. 
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Is every finitely generated torsion group finite?

Answered negatively by Golod in 1964.

Naive idea: Let \( F(x, y) = \{1, f_1, f_2, \ldots\} \) and \( G = \langle x, y \mid f_{1}^{n_1}, f_{2}^{n_2}, \ldots \rangle \).

Clearly \( G \) is torsion. Why is it non-trivial???

Theorem (Gromov-Olshanskii)

If \( G \) is non-elementary hyperbolic and \( g \in G \), then \( G/\langle\langle g^n \rangle\rangle \) is non-elementary hyperbolic for some \( n >> 1 \).

It follows that \( G_k = \langle x, y \mid f_{1}^{n_1}, \ldots, f_{k}^{n_k} \rangle \) can be made non-elementary hyperbolic for all \( k \). Therefore, \( |G| = \infty \).
Let $\pi(G)$ denote the set of orders of elements of a group $G$. 

Theorem (Osin, 2010) Every countable group $H$ can be embedded in a finitely generated group $G$ such that $\pi(G) = \pi(H) \cup \{\infty\}$ and all elements of the same order in $G$ are conjugate. 

Applying to $H = \mathbb{Z}$, we obtain: 

Corollary There exists finitely generated group other than $\mathbb{Z}/2\mathbb{Z}$ with 2 conjugacy classes. 

The proof uses small cancellation theory in relatively hyperbolic groups. It was generalized to acylindrically hyperbolic groups by Hull (2016) and new applications were found by Hull and Hull-Osin.
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Open questions

1. (Geometric rigidity) Is the class of finitely generated a.h. groups closed under quasi-isometry? It is even unknown if a finite extension of an a.h. group is a.h.

2. (Analytic vs geometric negative curvature) Assume that $G$ is torsion free and $\beta(2)(G) > 0$. Is $G$ a.h.? False for torsion groups (Lück-Osin, 2011).

Conjecture
If $q: G \rightarrow \ell^2(G)$ is a cocycle, then $H = \{ g \in G | q(g) = 0 \} \hookrightarrow \rightarrow hG$.

3. (Model-theoretic rigidity) Assume that $G$ is a.h., $H$ is finitely generated, and $\text{Th}(G) = \text{Th}(H)$. Is $H$ a.h.?
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