Acylindrically hyperbolic groups

D. Osin

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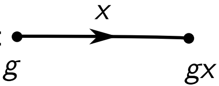
$V(\Gamma) = G, \qquad E(\Gamma):$



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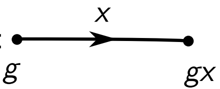

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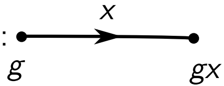
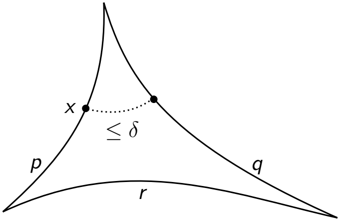
   

2. Let S be a metric space and let $G \curvearrowright S$ (by isometries). Fix $s \in S$ and study the geometric structure of the G-orbit of s.

These approaches work especially well in the presence of certain "negative curvature" conditions.

Definition (Gromov)

A geodesic metric space is *hyperbolic* if $\exists\,\delta \geq 0$ such that for any triangle with geodesic sides p, q, r and any $x \in p$, we have $\mathrm{d}(x, q \cup r) \leq \delta$.



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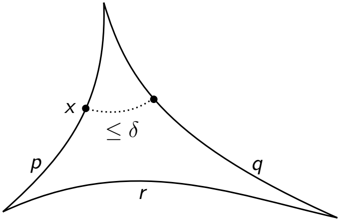

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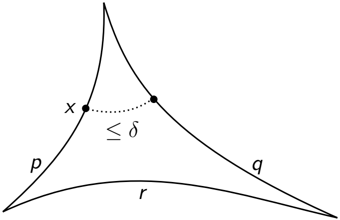

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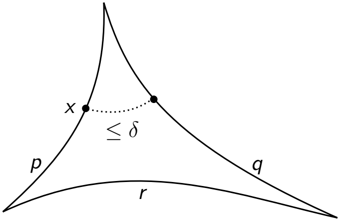

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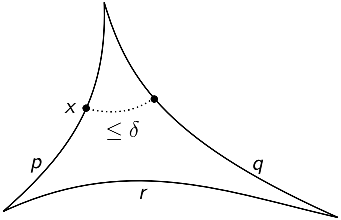

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2. Any tree is hyperbolic with $\delta = 0$.
3. $\mathbb{H}^n$ is hyperbolic.
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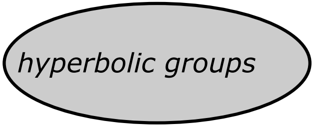

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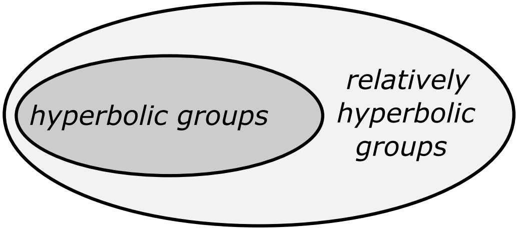

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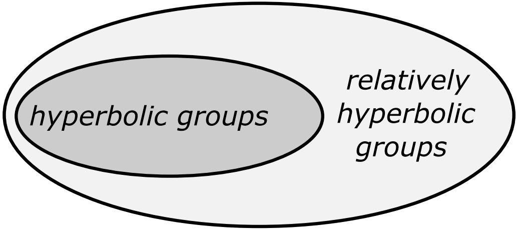

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An isometric action of G on a metric space S is acylindrical if $\forall\, \varepsilon > 0\ \exists\, R, N > 0$ such that $\forall\, x, y \in S$

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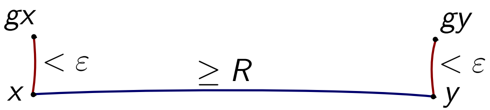

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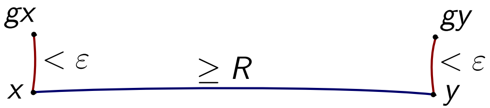

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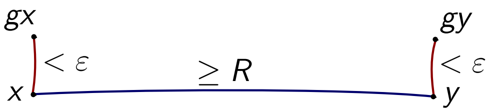

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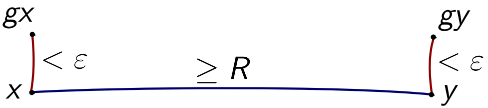

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- $G \curvearrowright pt$.
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- $MCG(S_g) \curvearrowright$ curve complex for $g \geq 2$ (Bowditch).

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- Groups of deficiency ≥ 2 (Osin).

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2. **Small cancellation theory:** proving embedding theorems and constructing groups with interesting properties.
3. **Measure theoretic rigidity:** every a.h. group G has plenty of quasi-cocycles $G \to \ell^2(G)$ (Hamenstadt, Hull-Osin, Bestvina-Bromberg-Fujiwara). It follows that $H^2_b(G, \ell^2(G)) \neq 0$ and Monod-Shalom rigidity theory for measure preserving actions applies.

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M is a compact 3-manifold, $\partial M = \mathbb{T}^2$. Fix $s \in \pi_1(\partial M)$ and let

$$M(s) = M \cup_\phi (\mathbb{S}^1 \times \mathbb{D}^2),$$

where $\phi \colon \partial(\mathbb{S}^1 \times \mathbb{D}^2) \to \partial M$ is such that $\phi(\partial \mathbb{D}^2) \in s$.

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Group theoretic Dehn filling

M	a.h. group G
∂M	hyperbolically embedded $H \leq G$
$s \in \pi_1(\partial M)$	$h \in H$
$M(s)$	$G/\langle\langle h \rangle\rangle^G$

Definition (Dahmani-Guirardel-Osin)

A subgroup $H \leq G$ is *hyperbolically embedded* (written $H \hookrightarrow_h G$) if there is a generating set X of G such that $Cay(G, X \cup H)$ is hyperbolic and satisfies a certain finiteness condition.

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3. If G is finitely generated and a.h., then a random subgroup is virtually free and hyperbolically embedded (Maher-Sisto).

Theorem (Dahmani-Guirardel-Osin, 2016)

Let $H \hookrightarrow_h G$. Then there exists finite $\mathcal{F} \subseteq H \setminus \{1\}$ such that for all $N \lhd H$ satisfying $N \cap \mathcal{F} = \emptyset$, we have

1. $\langle\langle N \rangle\rangle^G \cap H = N$; *equivalently, the natural map* $H/N \to G/\langle\langle N \rangle\rangle^G$ *is injective.*
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Remarks.

— First proved for relatively hyperbolic groups by Osin, 2007. Independent proof for torsion free relatively hyperbolic groups by Groves-Manning, 2008.

— Implies Thurston's theorem (modulo the geometrization conjecture).

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3. Structure of the kernel:

Theorem (Dahmani-Guirardel-Osin, 2016)

Under the assumptions of the main theorem, $\langle\langle N\rangle\rangle^G$ is isomorphic to the free product of copies of N.

Implies solution of two open problems about mapping class groups.

Constructing groups with interesting properties

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It follows that $G_k = \langle x, y \mid f_1^{n_1}, \ldots, f_k^{n_k} \rangle$ can be made non-elementary hyperbolic for all k. Therefore, $|G| = \infty$.

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Every countable group H can be embedded in a finitely generated group G such that $\pi(G) = \pi(H) \cup \{\infty\}$ and all elements of the same order in G are conjugate.

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The proof uses small cancellation theory in relatively hyperbolic groups. It was generalized to acylindrically hyperbolic groups by Hull (2016) and new applications were found by Hull and Hull-Osin.

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2. (Analytic vs geometric negative curvature) Assume that G is torsion free and $\beta_1^{(2)}(G) > 0$. Is G a.h.?

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2. (Analytic vs geometric negative curvature) Assume that G is torsion free and $\beta_1^{(2)}(G) > 0$. Is G a.h.? False for torsion groups (Lück-Osin, 2011).

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3. (Model-theoretic rigidity) Assume that G is a.h., H is finitely generated, and $Th(G) = Th(H)$. Is H a.h.?