Complex Brunn-Minkowski Theory.

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First version of Brunn-Minkowski, BM 1:

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Theorem

Let A_0 and A_1 be two convex bodies in $\mathbb{R}^n$ and let for $0 \le t \le 1$

$$A_t = tA_1 + (1-t)A_0 = \{ta_1 + (1-t)a_0; a_1 \in A_1, \quad a_0 \in A_0\}.$$

Then the volumes of A_t satisfy

$$|A_t|^{1/n} \ge t|A_1|^{1/n} + (1-t)|A_0|^{1/n}.$$

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In other words,

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is a concave function of t.

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Let $\mathcal{A}$ be a convex body in $\mathbb{R}^{n+1}$ and let

$$A_t = \{x \in \mathbb{R}^n; (t, x) \in \mathcal{A}\}$$

Then $t \to \log |A_t|$ is a concave function of t.

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Here we shall discuss versions of **BM 2** for complex convexity, i. e. plurisubharmonic functions and pseudoconvex domains or manifolds.

First we state a version of **BM 2** for functions:

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Let $\phi(t,x)$ *be a convex function on* $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x$. *Put*

$$\tilde{\phi}(t) := -\log \int_{\mathbb{R}^n} e^{-\phi(t,x)} d\lambda(x),$$

so that

$$e^{-\tilde{\phi}(t)} = \int_{\mathbb{R}^n} e^{-\phi(t,x)} d\lambda(x).$$

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2. Can be proved (Brascamp-Lieb) using an L^2-estimates for the equation $du = f$.

3. If $\tilde{\phi}$ is linear, $\phi(t,x) = \Phi(x + t\vec{v}) + at$ for some vector $\vec{v}$ in $\mathbb{R}^n$ and some a in $\mathbb{R}$.

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Kiselman's example: Let

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ϕ is plurisubharmonic in $\mathbb{C}^2$. But

$$\tilde{\phi}(\tau) = -\log \int_C e^{-\phi(\tau,z)} d\lambda(z) = -|\tau|^2 + C$$

is not plurisubharmonic.

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So, we replace the constant 1 (element in Ker(d)) by a holomorphic function (element in Ker($\bar{\partial}$)).

More generally:

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Let $\mathcal{D}$ be a pseudoconvex domain in $\mathbb{C}^{n+1}$. Let, for $\tau \in \mathbb{C}$,

$$D_\tau := \{z \in \mathbb{C}^n; (\tau, z) \in \mathcal{D}\}.$$

Let ϕ be plurisubharmonic in $\mathcal{D}$. Put

$$A^2_\tau := \{h \in H(D_\tau); \int_{D_\tau} |h|^2 e^{-\phi(\tau,z)} d\lambda(z) := \|h\|^2_\tau < \infty\},$$

the associated 'Bergman space' of holomorphic functions on D_τ.

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E, with $E_\tau = A^2_\tau$.

A section $\tau \to h_\tau$ is holomorphic if $h_\tau(z)$ is holomorphic in (τ, z).

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If the bundle were of finite rank, we could think of the norms as a matrix valued function $\tau \to H(\tau)$. Then positive curvature means that

$$-\frac{\partial}{\partial\bar\tau}H^{-1}\frac{\partial}{\partial\tau}H \geq 0,$$

i.e. '$-\log H$' is subharmonic.

Precise version; Theorem A:

Theorem

Let $\tau \to \mu_\tau$ be measures, compactly supported in D_τ with the property that

$$\tau \to \int_{D_\tau} h(\tau, z) d\mu_\tau(z)$$

is a holomorphic function of τ if $h \in H(\mathcal{D})$. Then

$$\tau \to \log \|\mu_\tau\|$$

is subharmonic. ($\|\mu_\tau\|$ is the norm as a functional on A^2_τ.)

Taking μ_τ to be point masses in $z_\tau \in D_\tau$ we get

Corollary

Let $B_\tau(z,z)$ be the Bergman kernel for A^2_τ. Then

$$\log B_\tau(z,z)$$

is plurisubharmonic in $\mathcal{D}$.

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Notice that if A is a (convex) domain

$$\frac{1}{|A|}$$

is the Bergman kernel for the space of constants!

More general picture:

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Look at complex manifolds X and Y and holomorphic maps $p : X \to Y$. For $y \in Y$ we put $X_y = p^{-1}(y)$. To fix ideas, assume that p is a proper submersion, so that the fibers X_y are smooth complex and compact manifolds of dimension, say, n. Let $(L, e^{-\phi})$ be a semipositively curved line bundle on X.

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Our substitute for the Bergman spaces A^2_τ are the spaces

$$E_y = H^{n,0}(X_y, L),$$

the space of L-valued holomorphic $(n, 0)$-forms on X_y, with norm

$$\|u\|_y^2 = c_n \int_{X_y} u \wedge \bar{u} e^{-\phi_y}.$$

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Assume X is Kähler. Then E_y make up a bona fide vector bundle E of positive curvature (in the sense of Nakano).

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Finally, we turn to applications/illustrations:

Complex interpolation.

Theorem

(Cordero-Erausquin)

Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two complex norms on $\mathbb{C}^n$. Let $\|\cdot\|_t$ be the intermediate norms defined by the method of complex interpolation and let A_t be their unit balls. Then

$$\log|A_t|$$

is concave.

Families of complex manifolds.

Let $p : X \to Y$ be a proper holomorphic submersion. Our vector bundle E with fibers

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That $p_*(K_{X/Y})$ has positive curvature is a theorem of Fujita, Griffiths – it follows from Griffiths theory of variation of Hodge structures. Theorem B can be seen as a twisted version of the Fujita-Griffiths result. It has been generalized by Paun-Takayama to general surjective maps, generalizing work of Viehweg and others on 'weak positivity' of direct image sheaves.

Curves in the space of metrics on a line bundle

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Let $X = Z \times \Omega$, where Z is compact and $\Omega \subset \mathbb{C}$. Let $L \to Z$ be a holomorphic line bundle that we pull back to a bundle $\underline{L}$ on X.

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A metric on $\underline{L}$ can be seen as a (complex) curve of metrics on L. Theorem B can therefore be used to study the Mabuchi-Semmes-Donaldson space of positively curved metrics on a fixed line bundle L on a compact manifold Z.

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So our vector bundle E is a line bundle. A metric on $-K_Z$ is canonically identified with a volume form on Z. Theorem B then says that

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So our vector bundle E is a line bundle. A metric on $-K_Z$ is canonically identified with a volume form on Z. Theorem B then says that

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Notice how similar this is to Prékopa's theorem!

This has been used to give a generalization of the Bando-Mabuchi uniqueness theorem for Kähler-Einstein metrics

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and Matsushima's theorem on the reductivity of automorphism groups (by Chen-Donaldson-Sun).

Estimates of Bergman kernels and Suita's conjecture

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$$D_s := \{z \in D, G(z) < s\}$$

Let B_s be the Bergman kernel at the origin for D_s.

Now, $\log B_s$ is convex in s and is asymptotic to $-ns$ when s approaches $-\infty$.

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This gives a proof of Suita's conjecture, first shown by Blocki and Guan-Zhou.

Thanks for your attention!