An invitation to higher Teichmüller theory

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ICM 2018
Rio de Janeiro, Brasil
The circle

The real line \((\mathbb{R}, <)\)

covers the circle and induces a cyclic order

\((x, y, z)\) is a positive triple
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Recast this ordering on \(\mathbb{RP}^1\)

Assume \(x = \mathbb{R}e_2, z = \mathbb{R}e_1\)

every \(y \neq z\) can be written as \(y = \begin{pmatrix} 1 & t_y \\ 0 & 1 \end{pmatrix} \cdot e_2\) with \(t_y \in \mathbb{R}\).
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The triple \((x, y, z)\) is positive if and only if \(t_y \in \mathbb{R}_{>0}\).
Filling the circle with life
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The circle is the boundary of the Poincaré disk.

The group of isometries is

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, |a|^2 - |b|^2 = 1 \right\} \cong SL(2,\mathbb{R})$$

acting by fractional linear transformations.
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The symmetry group of the tiling is a discrete subgroup, the quotient is a two dimensional hyperbolic orbifold.
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Hyperbolic structures on $S$ \hspace{1cm} discrete subgroups of $\text{PSL}(2, \mathbb{R})$ isomorphic to $\pi_1(S)$
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Hyperbolic structures on \( S \)

\( \text{Hyp}(S) \)

Discrete subgroups of \( \text{PSL}(2,\mathbb{R}) \)

Isomorphic to \( \pi_1(S) \)

\( \text{Hyp}(S) \subset \text{Hom}(\pi_1(S), \text{PSL}(2,\mathbb{R}))/\text{PSL}(2,\mathbb{R}) \)
The space of hyperbolic structures \( \text{Hyp}(S) \subset \text{Hom}(\pi_1(S), \text{PSL}(2,\mathbb{R}))/\text{PSL}(2,\mathbb{R}) \) is a connected component consisting entirely of discrete and faithful representations.

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$\text{Hyp}(S)$ can be identified with the Teichmüller space of $S$.

Representations in $\text{Hyp}(S)$ are positive representations.

The action of $\rho(\pi_1(S))$ on the circle is by orientation preserving homeomorphism.

For every representation $\rho \in \text{Hyp}(S)$ there is an equivariant map $\xi : \mathbb{R}P^1 \cong \partial \pi_1(S) \to \mathbb{R}P^1$ sending positive triples to positive triples.
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Representations in \( \text{Hyp}(S) \) are those of maximal Euler number

The Euler number is the obstruction to lifting to universal covering \( \widetilde{\text{PSL}(2, \mathbb{R})} \)

\[ \rho : \pi_1(S) = \{ a_1, b_1, \cdots a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \} \to \text{PSL}(2, \mathbb{R}) \]

\[ \text{eu}(\rho) = \widetilde{A_1} \widetilde{B_1} \widetilde{A_1}^{-1} \widetilde{B_1}^{-1} \cdots \widetilde{A_g} \widetilde{B_g} \widetilde{A_g}^{-1} \widetilde{B_g}^{-1} \in \mathbb{Z} \cap [2 - 2g, 2g - 2] \]  

[Goldman]
Higher Teichmüller spaces

Replace $\text{PSL}(2,\mathbb{R})$ by a Lie group $G$ of higher rank

A higher Teichmüller space is a union of connected components of $\text{Hom}(\pi_1(S), G)/G$ consisting entirely of discrete and faithful representations.
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Hitchin component for $n=3$ is the space of convex real projective structures on $S$

[Goldman, Choi-Goldman]
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Hitchin component for $n=3$ is the space of convex real projective structures on $S$

$\text{Hit}(S, \text{SL}(n, \mathbb{R}))$

is the connected component containing the irreducible Fuchsian representation $\pi_1(S) \to \text{SL}(2, \mathbb{R}) \to \text{SL}(n, \mathbb{R})$

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$\text{Max}(S, \text{Sp}(2n, \mathbb{R}))$

is the set of representations of maximal Toledo number, where $T$ is the obstruction to lifting to $\text{Sp}(2n, \mathbb{R})$

$$T(\rho) = \widetilde{A_1 B_1 A_1^{-1} B_1^{-1}} \cdots \widetilde{A_g B_g A_g^{-1} B_g^{-1}} \in [(2-2g)n, (2g-2)n]$$

[Goldman, Choi-Goldman]
Old and new features

Higher Teichmüller spaces share many features with \( \text{Hyp}(S) = \text{Teich}(S) \)

\[
\text{Hit}(S, G) \cong \mathbb{R}^{(2g-2)\dim(G)} \quad \text{[Hitchin]}
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New interesting features arise —

Arithmetics there are infinitely many integer points in the Hitchin component [Long-Reid-Thistlethwaite, Burger-Labourie-W]

Topology the space of maximal representations has non trivial topology [Gothen, GarciaPrada-Gothen-Mundet]

For \( \text{Sp}(4, \mathbb{R}) \) there are components where every representation is Zariski dense [Guichard-W, Bradlow-GarciaPrada-Gothen]
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**Hitchin component**  Internal parameters and dynamics  **Maximal representations**  noncommutative cluster algebras
Fenchel-Nielsen parametrization

Fenchel-Nielsen coordinates \( \text{Hyp}(S) \cong \mathbb{R}^{6g-6} \)

Key point: hyperbolic structure determined by length of the three boundary curves
Fenchel-Nielsen parametrization

Fenchel-Nielsen coordinates \( \text{Hyp}(S) \cong \mathbb{R}^{6g-6} \)

\[(l_i, \tau_i)_{i=1,\ldots,3g-3}\]

Key point: hyperbolic structure determined by length of the three boundary curves

Natural deformations: Fenchel-Nielsen twist flows

There is a natural symplectic structure on \( \text{Hom}(\pi_1(S), \text{PSL}(2,\mathbb{R}))/\text{PSL}(2,\mathbb{R}) \) [Goldman]

Twist flows are the Hamiltonian flows of the length functions

Wolpert’s formula \( \omega = \sum_{i=1}^{3g-3} dl_i \wedge d\tau_i \) completely integrable system
Dynamics on Hitchin component

Internal parameters
[Goldman, Fock-Goncharov, Bonahon-Dreyer, Zhang]

cut a pair of pants
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\[ \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup T \]

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\[ \Omega_t = \phi_t(\Omega) = g_1(t)\Omega_1 \cup g_2(t)\Omega_2 \cup g_3(t)\Omega_3 \cup T(t) \]

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g_1(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{t}{3}} & 0 \\ 0 & 0 & e^{-\frac{t}{3}} \end{bmatrix}, \quad g_2(t) := \begin{bmatrix} e^{-\frac{t}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{t}{3}} \end{bmatrix}, \quad g_3(t) := \begin{bmatrix} e^{\frac{t}{3}} & 0 & 0 \\ 0 & e^{-\frac{t}{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}
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twist flows + eruption flows of pants give completely integrable system on \( \text{Hit}(S, \text{SL}(n, \mathbb{R})) \)

[W-Zhang, Sun-W-Zhang, Sun-Zhang]
Cluster algebras

Parametrizations via ideal triangulations when $S$ has punctures/boundary

shear coordinates — X coordinates
decorated representations: fix a line $l \in \mathbb{RP}^1$ for each boundary curve

\[
\begin{align*}
x^1 & \quad x^0 \\
x^2 & \quad x^3 \\
x^4 & \quad x^0
\end{align*}
\]

\[
\begin{align*}
x^1(1 + x^0) & \quad x^2(1 + (x^0)^{-1})^{-1} \\
x^4(1 + (x^0)^{-1})^{-1} & \quad x^3(1 + x^0)
\end{align*}
\]

$x^i$ cross-ratios

lamba-lengths — A coordinates
framed representations: fix in addition a vector $v \in l \in \mathbb{RP}^1$ (horocycle)

\[
\begin{align*}
a_1 & \quad a_2 \\
a_0 & \quad a_0 \\
a_4 & \quad a_3
\end{align*}
\]

\[
\begin{align*}
a_1a_3 + a_2a_4 & \quad a_1
\end{align*}
\]

$a_i$ distances/determinant

[Thurston, Penner, Fock-Goncharov]
Noncommutative cluster algebras

Parametrizations of maximal representations via ideal triangulations of $S$ give a geometric realization of noncommutative cluster algebras [Berenstein-Retakh]

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$X$ coordinates

decorated representations: for each boundary curve a fixed Lagrangian $L \in \text{Lag}(\mathbb{R}^{2n})$

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x^i \to X^i \in \text{Pos}(n, \mathbb{R})
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**A coordinates**

framed representations: over each puncture fix a basis $B$ of $L \in \text{Lag}(\mathbb{R}^{2n})$

\[
a_i \to A_i \in \text{GL}(n, \mathbb{R})
\]

\[
a_1 a_3 + a_2 a_4 \quad a_0
\]

\[
\frac{a_1 a_3 + a_2 a_4}{a_0} := A_1 A_0^{-1} A_3 + A_4 A_0^{-1} A_2
\]
Common features

Hitchin representations and maximal representations ...

... are examples of Anosov representations

... are holonomy representations of geometric structures on compact manifolds

... are positive representations
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\[ \rho \in \text{Hit}(S, SL(n, \mathbb{R})) \] if and only if there exists an equivariant map \( \xi : \mathbb{RP}^1 \cong \partial \pi_1(S) \to \text{Flag}(\mathbb{R}^n) \) sending positive triples to positive triples of flags. [Labourie, Guichard, Fock-Goncharov]
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\( (L^+, L, L^-) \) is positive if and only if \( L = \begin{pmatrix} 1 & P_L \\ 0 & 1 \end{pmatrix} \cdot L^+ \) for a positive definite \( P_L \)

\[ \text{[Burger-lozzi-W]} \]
Positivity

The two notions of positivity stem from Lusztig’s total positivity and from order positivity in Hermitian Lie groups. They have in fact a common generalization.

In both cases the set of positive triples is a connected component of the intersection of two open Bruhat cells

$$\Omega_{F^+} \cap \Omega_{F^-}$$

and this connected component has the structure of a semigroup.

Example: $\text{SL}(2, \mathbb{R})$ acting on the circle

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A simple Lie group \( G \) admits a positive structure in a generalized flag variety \( \mathcal{F} = G/P \) if for every pair of transverse flags \((F^+, F^-)\) there is a connected component of the intersection \( \Omega_{F^+} \cap \Omega_{F^-} \), which has the structure of a semigroup.
## New positive structures

There are four families of Lie groups which admit a positive structure:

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Positivity given by semigroup in the unipotent radical of the parabolic subgroup. There is a positive semigroup in $G^0 < G$. The elements in $G^0$ act proximal on $\mathcal{F}$. 

[Guichard-W]
New positive structures

There are four families of Lie groups which admit a positive structure:

- **$G$** split real group \( \mathcal{F} = G/B \)  full flag variety  Lusztig’s total positivity
- **$G$** Hermitian type \( \mathcal{F} = G/Q \)  Shilov boundary  order positivity
- **$G$** $\text{SO}(p, q), p < q$ \( \mathcal{F} = \mathcal{F}_{1, \ldots, p-1} \)  partial flag variety
- **$G$** with root system $F_4$ \( \mathcal{F} = \mathcal{F}_{\alpha_1, \alpha_2} \)  partial flag variety

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Positivity given by semigroup in the unipotent radical of the parabolic subgroup. There is a positive semigroup in $G^>0 < G$.
The elements in $G^>0$ act proximal on $\mathcal{F}$.

Lusztig positivity is related to cluster algebras

— expect new noncommutative cluster algebras
Positive representations

A representation $\rho : \pi_1(S) \to G$ is positive if there exists an equivariant map

$$\xi : \mathbb{RP}^1 \cong \partial \pi_1(S) \to \mathcal{F}$$

that sends positive triple to positive triples of flags.

Examples:

$$\rho : \pi_1(S) \to \text{SL}(2,\mathbb{R}) \to \text{SO}(p, p-1) \to \text{SO}(p, q)$$

$$\rho : \pi_1(S) \to \text{SL}(2,\mathbb{R}) \to \text{SO}(p, p+1) \to \text{SO}(p, q)$$
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Every positive representations is discrete and faithful ($\mathcal{F} - \text{Anosov}$).
The set of positive representations is open.

[Guichard-Labourie-W]
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[Guichard-Labourie-W]

The set of positive representations is closed among irreducible representations.  

[Guichard-Labourie-W]
There are higher Teichmüller spaces $T_{pos}(S, G) \subset \text{Hom}(\pi_1(S), G)/G$ if and only if $G$ admits a positive structure.

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Conjectures/Predictions

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The second prediction has been partly confirmed for \( \text{SO}(p, q) \) using methods from the theory of Higgs bundles.

[Collier, Aparicio-Arroyo-Bradlow-Collier-García-Prada-Gothen-Oliveira]

[Guichard-Labourie-W]
Conclusion

Higher Teichmüller spaces were discovered from very different view points. Positivity is the common underlying structure. We conjecture it to be the reason for the existence of higher Teichmüller spaces.
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What we see is only the tip of the iceberg. There is a lot waiting to be discovered beneath the surface!

Get on your diving gear and explore yourself!