Degenerations and moduli spaces in Kähler geometry

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Plan

- Background on Kähler manifolds
- Motivations from Riemann surfaces
- Kähler-Einstein metrics on Fano manifolds
- Further developments

Background

A Kähler metric on a smooth manifold X^{2n} consists of

- a Riemannian metric g, and
- a complex structure J

which satisfy the compatibility conditions

$$\begin{cases} g(J\cdot, J\cdot) = g(\cdot, \cdot) \\ \\ \nabla J = 0 \end{cases}$$

Central theme:

g (Differential geometry)

$$\updownarrow$$

J (complex/algebraic geometry)

On the one hand, g is more intrinsic:

- The Kähler condition can be interpreted as the holonomy group of g being contained in $U(n) = SO(2n) \cap GL(n;\mathbb{C})$.
- For Kähler metrics, J is essentially determined by g (if g is locally not a product and not hyperkähler);

On the other hand, working on a fixed compact complex manifold (X, J), Kähler metrics can be described in terms of complex geometric data

- The Kähler form $\omega = g(J\cdot, \cdot)$ is locally given by

$$\omega = \sqrt{-1}\partial\bar{\partial}\phi := \sqrt{-1}\sum_{\alpha,\beta} \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta$$

  for a strictly pluri-subharmonic function ϕ.

- We globally get a Kähler class $[\omega]$ in $H^2(X; \mathbb{R})$.

- Any other Kähler form ω' in $[\omega]$ is given by

$$\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi$$

for a global function ϕ.

- Example: $X \subset \mathbb{CP}^N$ a smooth algebraic subvariety,

$$\omega = \sqrt{-1}\partial\bar{\partial}\log|z|^2$$

One can get different Kähler forms in the same cohomology class by replacing z with $A.z$ for $A \in GL(N+1;\mathbb{C})$.

A basic question is to study canonical metrics on a given compact complex manifold X.

Generally the notion of being “canonical" is defined in terms of certain curvature conditions

In particular, we are interested in various aspects, including

- Existence
- Uniqueness
- Degenerations and moduli

These questions are connected to algebraic/complex geometry and are also inter-related themselves.

Riemann Surfaces

Consider a closed Riemann surface X of genus $g > 1$.

By the uniformization theorem, X admits a unique conformal metric of constant curvature -1.

This is (indirectly) related to the algebro-geometric fact:

X can be realized as a stable algebraic curve in a projective space $\mathbb{CP}^N$ in the sense of geometric invariant theory.

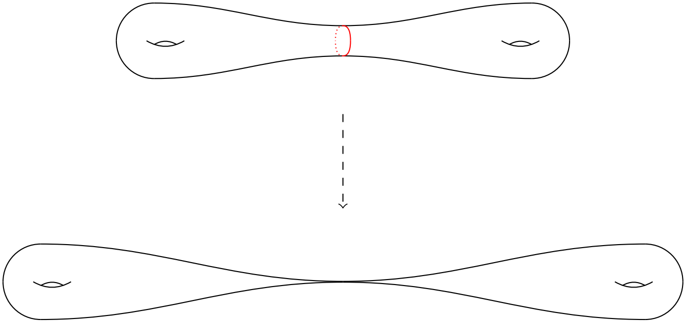
- Local moduli count:

$6g - 6$ real parameters in terms of hyperbolic metrics

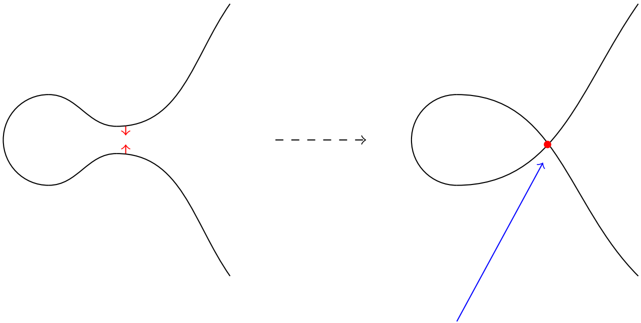
$3g - 3$ complex parameters in terms of complex structures

- Globally, degenerations can occur.
- Metric degenerations (in the sense of Gromov-Hausdorff) and stable curve degenerations (in the sense of Deligne-Mumford) are compatible.

In terms of hyperbolic metrics, the diameter can go to infinity and hyperbolic cusps can appear:



In terms of stable curves, nodal singularities can form:



locally $\{x^2 + y^2 = 0\} \subset \mathbb{C}^2$

Higher dimensions:

Kähler-Einstein metrics

Let X be a compact complex manifold. A Kähler metric ω is Kähler-Einstein if

$$Ric(\omega) = \lambda\omega, \quad \lambda \in \mathbb{R}$$

Local form:

$$\det\left(\frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}\right) = e^{-\lambda\phi + f}$$

Necessary condition:

$$2\pi c_1(X) = \lambda \cdot [\omega] \qquad \text{in } H^2(X; \mathbb{R})$$

- More generally, we can study the equation

$$\Delta_\omega Ric(\omega) = 0$$

which is equivalent to ω having constant scalar curvature.

- These are special cases of Calabi's extremal Kähler metrics, which are critical points of the following functional

$$\tilde{\omega} \in [\omega] \mapsto \int_X |Riem(\tilde{\omega})|^2 \tilde{\omega}^n$$

Calabi conjecture (1954):

There is a Kähler-Einstein metric in the Kähler class $[\omega]$ *if and only if* $c_1(X)$ *is proportional to* $[\omega]$.

Theorem (Aubin 1976, Yau 1976): Calabi Conjecture is true when $c_1(X) < 0$;

Theorem (Yau 1976): Calabi Conjecture is true when $c_1(X) = 0$;

In both cases, the Kähler-Einstein metric is unique in $[\omega]$.

When $c_1(X) > 0$, X is called a Fano manifold. We have an algebro-geometric characterization of the existence question:

Theorem (Chen-Donaldson-Sun 2012):

X admits a Kähler-Einstein metric $\Longleftrightarrow$ X is K-stable.

The “only if" direction was due to Tian, Donaldson, Stoppa, Mabuchi, Berman...

In this case, if the Kähler-Einstein metric exists, it is unique modulo the action of $Aut(X)$ (Bando-Mabuchi 1987).

This proves a special case (when X is Fano and $L = K_X^{-1}$) of the

Yau-Tian-Donaldson Conjecture:

Let X be a compact complex manifold and L an ample line bundle over X. Then there is a constant scalar curvature Kähler metric $\omega \in 2\pi c_1(L)$ if and only if (X, L) is K-stable.

K-stability

K-stability is an algebro-geometric notion (Tian 1997, Donaldson 2002).

It generalizes the Hilbert-Mumford criterion for stability in geometric invariant theory.

It involves two key concepts:

Test Configurations and the Futaki invariant.

A Test Configuration $\mathcal{X}$ is a degeneration of (X, L) given by

- a projective embedding of $X \subset \mathbb{CP}^N$ using sections of L^r for some r, and
- a one parameter subgroup

$$\lambda : \mathbb{C}^* \to PGL(N+1; \mathbb{C})$$

Taking the limit of $\lambda(t).X$ as $t \to 0$, we obtain an algebraic scheme (X_0, L_0).

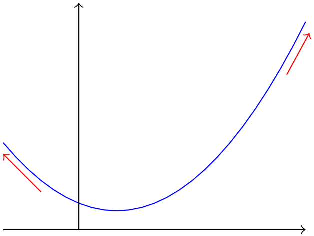
The Futaki invariant $Fut(\mathcal{X})$ is a number associated to $\mathcal{X}$.

It can be calculated in terms of the weights of the $\mathbb{C}^*$ action on the vector spaces $H^0(X_0, L_0^k)$ for $k \gg 1$.

Definition: (X, L) is K-stable if $Fut(\mathcal{X}) \geq 0$ for all test configurations and "$=$" occurs only in trivial cases.

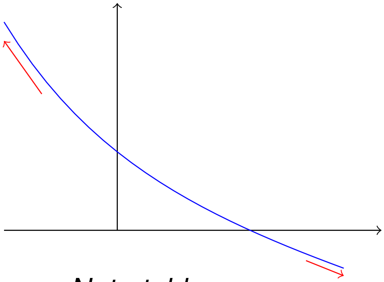
There are many variations of the notion of K-stability, on different levels of strength.

Heuristic picture: convex functions on $\mathbb{R}$



Stable

Critical point exists



Not stable

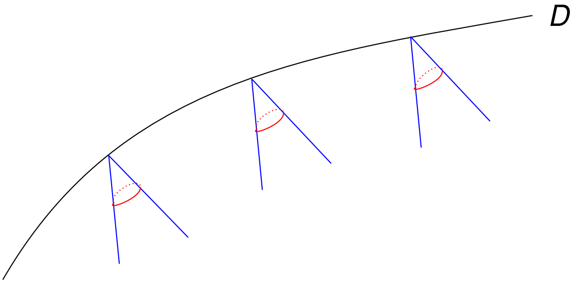
No critical points

Ideas in the proof

Main strategy is by the continuity method.

- Start with a chosen initial Kähler metric
- Deform it towards a Kähler-Einstein metric via a prescribed process
- If the deformation breaks down at some point, then we produce a de-stabilizing test configuration.

More precisely, we consider Kähler-Einstein metrics with cone singularities along a smooth divisor D, defined by a holomorphic section of K_X^{-a} for some $a > 1$.



Families of equations:

$$Ric(\omega_\beta) = \mu_\beta \omega_\beta + 2\pi(1-\beta)[D], \quad \beta \in (0,1]$$

• The set of $\beta \in (0,1]$ such that the above equation is solvable is open (Donaldson 2011)

• $\mu_\beta < 0$ for $0 < \beta = 1/p \ll 1$, which is the starting point

• $\mu_\beta = 1$ when $\beta = 1$, which yields the desired Kähler-Einstein metric

A key step is to obtain algebro-geometric structures on the limits when $\beta_i \to \beta_\infty$ but ω_{β_i} does not converge.

This is related to a simpler question concerning degenerations of smooth Kähler-Einstein metrics.

Given X_i Fano, $\dim_{\mathbb{C}} X_i = n$, $Ric(\omega_i) = \omega_i$.

Passing to a subsequence we obtain a Gromov-Hausdorff limit, which is a compact metric space.

Geometric bounds in our setting:

- $Diam(X_i, \omega_i) \leq D$ (Myers theorem)

- Volume non-collapsing (Bishop-Gromov theorem)

$$Vol(B(p,1)) \geq \epsilon > 0, \quad \forall\, p \in X_i, \ \forall \ i.$$

This contrasts with the negatively curved and Ricci-flat cases.

The Cheeger-Colding regularity theory, in this non-collapsing case, yields:

Any Gromov-Hausdorff limit X_∞ admits a decomposition

$$X_\infty = \mathcal{R} \cup \mathcal{S}.$$

The regular set $\mathcal{R}$ is a smooth Kähler-Einstein manifold;

The singular set $\mathcal{S}$ is a closed subset which is of Hausdorff co-dimension at least 2 (improved to 4 by Cheeger-Colding-Tian).

Theorem (Donaldson-Sun 2012)

- *X_∞ naturally carries a structure of a normal projective algebraic variety.*
- *The metric singular set $\mathcal{S}$ = the algebraic singular set.*

The proof here depends crucially on techniques from complex geometry (L^2 estimate).

Note: The structure of $\mathcal{S}$ in the general Riemannian setting remains an interesting open question.

Theorem (Donaldson-Sun 2012)

There exist $k = k(n), \epsilon = \epsilon(n) > 0$ such that for all i, and for all $x \in X_i$,

$$\rho_{k,\omega_i}(x) \geq \epsilon.$$

Here

$$\rho_{k,\omega_i}(x) = \sup\{|s(x)|^2 \mid s \in H^0(X_i, K_{X_i}^{-k}), \int_{X_i} |s|^2 = 1\}$$

is the Bergman function, which controls the geometry of the L^2 embedding of X_i into the projective spaces.

- This result proves a conjecture of G. Tian ICM 1990.

- In the negatively curved and Ricci-flat case, one obtains similar results as long as one assumes $[\omega_i]$ is integral and the diameter is bounded above.

  This extra condition has algebro-geometric meaning (c.f. Rong-Zhang, Tosatti, J. Song).

- There are generalizations to the case when we replace the Kähler-Einstein condition by only a lower bound on the Ricci curvature (Chen-Wang 2014, G. Liu-Székelyhidi 2018).

The proof of the Kähler-Einstein result (Chen-Donaldson-Sun 2012) relies on an extension to the case with cone singularities.

If the deformation process breaks down at some point β, then we need to

- Construct X_0;
- Construct a one parameter subgroup $\lambda(t)$;
- Show that the Futaki invariant is non-positive.

Other proofs of the Kähler-Einstein result:

- Classical continuity path (Datar-Székelyhidi 2015).
- Ricci-flow (Chen-Sun-Wang 2015) [based on the proof of the Hamilton-Tian conjecture (Chen-Wang 2015)].

These two methods prove the result under the weaker assumption of equivariant K-stability, which is often checkable in explicit examples (c.f. Ilten-Süss 2015, Delcoix 2016)

- Variational approach (Berman-Boucksom-Jonsson 2015). This proves the result under the stronger assumption of uniform K-stability.

Further developments

More on singularities

We again consider a sequence of smooth Fano Kähler-Einstein metrics (X_i, ω_i), with a Gromov-Hausdorff limit Z.

Theorem (Donaldson-Sun 2015)

- *Rescaled Gromov-Hausdorff limits are affine algebraic varieties endowed with asymptotically conical Calabi-Yau metrics*
- *At any singular point $p \in Z$, there is a unique metric tangent cone $\mathcal{C}_p$.*

Asymptotically conical Calabi-Yau metrics have been extensively studied (Tian-Yau, Conlon-Hein, C. Li, G. Liu, ...)

The cone structure on $\mathcal{C}_p$ can be interpreted algebraically as a grading on its function ring

$$R(\mathcal{C}_p) = \bigoplus_{d \in \mathcal{D}} R_d$$

The grading induces a valuation on the ring $\mathcal{O}_p$ of germs of holomorphic functions near p:

$$d : \mathcal{O}_p \setminus \{0\} \to \mathcal{D}; \quad f \mapsto \lim_{r \to 0} \frac{\sup_{B(p,r)} \log |f|}{\log r}$$

The associated graded ring $R(p)$ is finitely generated, and it defines an algebraic cone

$$W_p = Spec(R(p))$$

Geometrically, W_p can be understood as the limiting object obtained via algebraic re-scaling, for some complex-analytic embedding of the singularity into an affine space $\mathbb{C}^m$ and some choice of weights

Furthermore, there is a $\mathbb{C}^*$ equivariant degeneration from W_p to $\mathcal{C}_p$ as algebraic cones

Both W_p and $\mathcal{C}_p$ are algebraic invariants of the singularity p (Conjectured in Donaldson-Sun 2015, Algebraized by C. Li 2016, Proved by Li-Liu 2016, Li-Xu 2017, Li-Wang-Xu 2018)

This is related to the volume minimization principle in Sasaki-Einstein geometry (Martelli-Sparks-Yau 2006)

Study of metric tangent cones has applications to a folklore conjecture in understanding conical behavior of singular Calabi-Yau metrics (Hein-Sun 2016)

A particularly important example is a nodal singularity locally modeled on

$$\{x_0^2 + \cdots + x_n^2 = 0\} \subset \mathbb{C}^{n+1}.$$

In this case, the singular metric is always asymptotic at a polynomial rate to the explicit Stenzel cone metric

$$\omega_S = \sqrt{-1}\partial\bar{\partial}(\sum_{j=0}^{n} |x_j|^2)^{\frac{n-1}{n}}.$$

Optimal degenerations

Ricci flow on Fano manifolds

$$\begin{cases} \frac{\partial}{\partial t}\omega_t = \omega_t - Ric(\omega_t), \quad t \in [0, \infty); \\ \omega_0 \in 2\pi c_1(X). \end{cases}$$

Chen-Sun-Wang 2015: If X is K-unstable, then given any initial metric ω_0, the Ricci flow solution uniquely determines a (possibly irrational) de-stabilizing test configuration.

W. He 2012, Dervan-Székelyhidi 2016: This test configuration is optimal in a suitable sense.

Moduli spaces

Taking Gromov-Hausdorff limits provides a topological compactification of the moduli space of Kähler-Einstein (K-stable) Fano manifolds.

Li-Wang-Xu 2014, Spotti-Sun-Yao 2014, Odaka 2014: There exists a natural algebraic structure on the compactified moduli space.

Explicit understanding of this moduli space is closely related to geometric invariant theory, and motivates interesting new questions (Mabuchi-Mukai 1993, Odaka-Spotti-Sun 2012, Spotti-Sun 2017, Liu-Xu 2017)

Example: $n = 2$, we consider cubic surfaces in $\mathbb{CP}^3$.

Odaka-Spotti-Sun 2012: The compactified Kähler-Einstein moduli $=$ the GIT moduli.

GIT stability of cubics was studied by Hilbert in 1893.

A point in the moduli represents

- either a cubic surface with at worst nodal singularities,
- or the cubic surface defined by the equation $\{xyz = t^3\}$.

Beyond the Fano case?

Fundamental open questions are to understand Riemannian convergence theory of constant scalar curvature Kähler metrics and the collapsing behavior.

There are progress in studying degeneration of Kähler-Einstein metrics, assuming results from algebraic geometry.

$c_1 < 0$ case:

There is a KSBA compactified moduli space in algebraic geometry, which is a higher dimensional analogue of the Deligne-Mumford compactification for curves. In the boundary we see varieties with certain mild singularities

The KSBA compactification can be viewed as the moduli of K-stable varieties (Odaka 2011)

Degenerations of Kähler-Einstein metrics are compatible with the KSBA degenerations (Berman-Guenancia 2014, J. Song 2017), generalizing the one dimensional example discussed earlier.

$c_1 = 0$ case:

Collapsing of Calabi-Yau metrics under complex structure degenerations remains a very interesting problem.

There is a related question of understanding limits of Calabi-Yau metrics when we fix the complex structure and let the Kähler class become degenerate, c.f. works of Tosatti and his collaborators.