Hodge Theory and Cycle Theory of Locally Symmetric Spaces

Tiling of $\mathcal{H}_3$ by right angled dodecahedra (J. Leys)
Cohomology of Discrete/Arithmetic Groups: motivations and heuristics.
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- General philosophy. The cohomology groups $H^\bullet(\Gamma)$ behave in many ways ‘like’ the cohomology groups of complex projective manifolds.
An arithmetic group $\Gamma$ is ‘the $\mathbb{Z}$-points of a classical group,’ e.g.

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The latter looks different but can be presented similarly as a group of $4 \times 4$ matrices with $\mathbb{Z}$-coefficients:

$$\begin{pmatrix} a + bi & c + di \\ e + fi & g + hi \end{pmatrix} \mapsto \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{pmatrix}$$
Why should you keep listening?

Because you like either:

- Non positively curved spaces and geometric group theory,
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- Non positively curved spaces and geometric group theory,
- Complex (algebraic) geometry,
- Number theory.
Symmetric spaces

- Each $\Gamma$ acts properly and discontinuously on a canonically associated contractible complete Riemannian manifold $S$ (‘symmetric space’).
- The locally symmetric space attached to $\Gamma$ is the quotient
  \[ X_\Gamma = \Gamma \backslash S. \]

It need not be compact, but it has finite volume (\(\sim\) ‘Complexity’ of $\Gamma$).
Most famous/basic examples

- If $\Gamma \subset \text{PSL}_2(\mathbb{R})$, the associated symmetric space $S$ is the Poincaré upper-half plane $\mathcal{H}_2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and the action of $\Gamma$ is by fractional linear transformations; it preserves the standard hyperbolic metric $|dz|^2/\text{Im}(z)^2$. The quotient $\Gamma \backslash S$ is a Riemann surface.
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I will forget about the complex structure and rather insist on the invariant (hyperbolic) metric!

- If $\Gamma \subset \text{PSL}_2(\mathbb{C})$, the associated symmetric space $S$ can be identified with the three-dimensional hyperbolic space $\mathcal{H}_3$, and $X_\Gamma$ is a finite volume hyperbolic 3-manifold.
Though $X_\Gamma$ is only a real manifold in general, many recent and not-so-recent works suggest that the groups

$$H^\bullet(\Gamma) = H^\bullet(X_\Gamma)$$

behave in many ways ‘like’ the cohomology groups of a complex projective manifold.
Hodge Theory of Locally Symmetric Spaces:

Matsushima, Vogan-Zuckerman.
Hodge Theory

For compact Riemannian manifold $M$

- Each cohomology class is represented by unique differential form $\omega$ of minimal $L^2$-norm, i.e.

  harmonic $k$-forms on $M \xrightarrow{\sim} H^k(M, \mathbb{C})$. 
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$$\text{harmonic } k\text{-forms on } M \xrightarrow{\sim} H^k(M, \mathbb{C}).$$

- If $M$ complex, there is a natural $S^1$-action on differential forms. Moreover if $M$ is Kähler, this $S^1$ preserves harmonic forms

$\sim$ ‘Hodge Theory’
From a Riemannian point of view, Kähler means that the metric has restricted holonomy — the complex structure on the tangent space is preserved by parallel transport.

Locally symmetric spaces also have highly restricted holonomy: there are algebraic structures on the tangent space that are invariant by parallel transport.

∼ extra algebraic structures on their cohomology
'Hodge Theory' for $X_{\Gamma}$ provides a canonical splitting

$$H^\bullet(\Gamma) = H^\bullet(X_{\Gamma})_{\text{inv}} \oplus (\ldots)$$

where

$H^\bullet(X_{\Gamma})_{\text{inv}}$: forms invariant under parallel transport,

$(\ldots)$ less and less invariant forms,

$\sim$: ‘filtration.’

The full algebra is more complicated; it is the theory of $(g, K)$-cohomology.
Consider the case of an irreducible compact quotient of

$$S = \mathcal{H}_3 \times \mathcal{H}_3 \times \mathcal{H}_3$$

by some discrete group

$$\Gamma \subset \text{PSL}_2(\mathbb{C})^3.$$
On Kähler manifolds, the $S^1$-action preserves harmonic forms. Here, analogously, we have a splitting according to the type of harmonic forms on each factor:

$$H^p(\Gamma, \mathbb{C}) = \bigoplus_{a+b+c=p} H^{a,b,c}.$$ 

We group terms to get:

$$H^\bullet(\Gamma) = \left( \bigoplus_{a,b,c \in \{0,3\}} H^{a,b,c} \right) \bigoplus \left( \bigoplus_{a,b,c \in \{1,2\}} H^{a,b,c} \right).$$
An example

\sim \text{analogue of the Hodge diamond.}

The vanishing of some $h^{a,b,c}$'s — e.g. $h^{1,0,0} = 0$ — puts a big constraint on the vector

$$(b_0, \ldots, b_9)$$

of Betti numbers: it has to be of the form

$$(1, 0, 0, 3, 0, 0, 3, 0, 0, 1) + m(0, 0, 0, 1, 3, 3, 1, 0, 0, 0)$$

for some integer $m \geq 0$. 
Betti numbers growth:

Locally Symmetric Spaces vs complex projective manifolds.
Asymptotics of Betti numbers

$M \subset \mathbb{P}^{n+1}$ hypersurface of degree $d$. $X_\Gamma$ locally symmetric manifold of volume $V$.

\[
\begin{align*}
b_k(M) &= O(1) \text{ if } k \neq n, \\
b_n(M) &= d^{n+1} + O(d^n).
\end{align*}
\]

\[
\begin{align*}
b_k(X_\Gamma) &= o(V) \text{ if } k \neq \frac{1}{2} \dim S, \\
b_{\frac{1}{2} \dim S}(X_\Gamma) &= \frac{\chi(S^c)}{\text{vol}(S^c)} V + o(V).
\end{align*}
\]
Accumulations of works ranging from DeGeorge and Wallach to the more recent ‘7 samurai’ work yield

**Theorem.** Suppose that $G$ has property (T) and rank at least two. The growth of the Betti numbers of locally symmetric $S$-manifolds is given by

$$b_k(X_\Gamma) = \begin{cases} 
\frac{\chi(S^c)}{\text{vol}(S^c)} \text{vol}(X_\Gamma) + o(\text{vol}(X_\Gamma)) & \text{if } k \neq \frac{1}{2} \dim S \\
o(\text{vol}(X_\Gamma)) & \text{if } k = \frac{1}{2} \dim S.
\end{cases}$$
Recent further works by Fraczyk and Raimbault yield:

– **Theorem.**– Let $S$ be arbitrary. The growth of the Betti numbers of congruence arithmetic $S$-manifolds is given by

$$b_k(X_\Gamma) = \begin{cases} 
 o\left(\text{vol}(X_\Gamma)\right) & \text{if } k \neq \frac{1}{2} \text{ dim } S \\
 \frac{\chi(S^c)}{\text{vol}(S^c)} \text{vol}(X_\Gamma) + o\left(\text{vol}(X_\Gamma)\right) & \text{if } k = \frac{1}{2} \text{ dim } S.
\end{cases}$$
Cycle theory of Locally Symmetric Spaces:

‘Hodge type Theorems.’
Hodge Conjecture

- $M = \text{complex projective manifold } M \subset \mathbb{P}^N \text{ over } \mathbb{C}$.
- $Z \subset M$ closed analytic subspace of co-dimension $p$.
- The class $\text{cl}(Z)$ in $H^{2p}(M, \mathbb{C})$ is of type $(p, p)$.
- Rational $(p, p)$-classes are called Hodge classes.

Hodge posed the famous:

- **Hodge Conjecture.** On a complex projective manifold, any Hodge class is a rational linear combination of cycle classes $\text{cl}(Z)$ of analytic subspaces.
Real hyperbolic manifolds

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E.g, for $r = 1$ and $n > 3$:

$$v \in \mathbb{Q}^{n+1} \rightsquigarrow \text{hyperplane } v^\perp = \{\ell \in \mathcal{H}_n \mid \ell \perp_Q v\}$$

$$\rightsquigarrow \text{co-dimension 1 sub-manifold in } M.$$

Theorem: Poincaré duals of these cycles span $H^1(M, Q)$, in fact even $H^1(M, \mathbb{C})$. 

• The tools used in the proof of the above ‘Hodge type theorem’ apply to many more locally symmetric spaces.

• They apply in particular to some important classes of complex projective manifolds.

– **Theorem (B.-Millson-Moeglin).**– On a projective unitary Shimura variety uniformized by the complex $n$-ball, any Hodge $(r, r)$-class with $r \in [0, n] \setminus \left[\frac{n}{3}, \frac{2n}{3}\right]$ is algebraic.
Lefschetz properties:

Oda, Harris, Li, Venkataramana, Clozel,...
The Hyperplane Section Theorem

- \( M = \) complex projective manifold \( M \subset P^N \) over \( \mathbb{C} \).
- \( H \subset P^N \) projective hyperplane transverse to \( M \).

Lefschetz proved:

- **Theorem.** The natural map

\[
H_q(H \cap M) \to H_q(M)
\]

is an isomorphism for \( q \leq \dim_{\mathbb{C}}(M) - 2 \) and surjective for \( q = \dim_{\mathbb{C}}(M) - 1 \).

- In fact, for every base point \(* \) in \( H \cap M \), the natural map

\[
\pi_q(H \cap M, *) \to \pi_q(M, *)
\]

is bijective for \( q \leq \dim_{\mathbb{C}}(M) - 2 \) and surjective for \( q = \dim_{\mathbb{C}}(M) - 1 \).
Lefschetz properties for locally symmetric spaces

- Replace hyperplane sections by locally symmetric sub-spaces.
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- A locally symmetric proper sub-space cannot span the homology: use Hecke translates! See Venkataramana’s ICM 2010 talk.
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- Injectivity: several partial results. Let’s mention the following homotopical version for hyperbolic manifolds.

- **Theorem (B.-Haglund-Wise).** A closed arithmetic hyperbolic manifold virtually retracts onto any of its closed co-dimension 1 totally geodesic sub-manifolds.
– Theorem (B.-Haglund-Wise).– A closed arithmetic hyperbolic manifold $\Gamma \backslash \mathcal{H}_n$ virtually retracts onto any of its closed co-dimension 1 totally geodesic sub-manifolds.

• Let $H \subset \mathcal{H}_n$ be a hyperplane s.t. $\text{Stab}_\Gamma(H) \backslash H$ is compact. There exists a finite index subgroup $\Gamma' \subset \Gamma$ and a morphism $r : \Gamma' \to \text{Stab}_{\Gamma'}(H)$ which is the identity when restricted to $\text{Stab}_{\Gamma'}(H)$. 
Homotopical Lefschetz properties for hyperbolic manifolds

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- In particular the induced map

$$H_q(\text{Stab}_{\Gamma'}(H) \backslash H) \to H_q(\Gamma' \backslash \mathcal{H}_n)$$

is injective for all $q \geq 0$. 

An example
A totally geodesic cycle
Retraction
Theorem (B.-Haglund-Wise). – A closed arithmetic hyperbolic manifold \( \Gamma \backslash \mathcal{H}_n \) virtually retracts onto any of its closed co-dimension 1 totally geodesic sub-manifolds.

- Use closed co-dimension 1 totally geodesic sub-manifolds to quasi-convexly embed a finite index subgroup \( \Gamma' \subset \Gamma \) into a right angled Coxeter group.
- Haglund: right angled Coxeter groups virtually retract onto their quasi-convex subgroups.
Period theory on Locally Symmetric Spaces: a conjecture.
Period matrices

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- Problem: bound the geometric complexity of cycles needed to generate homology.
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We encode this into period matrices:

\[
\begin{bmatrix}
\int_{\gamma_k} \omega_\ell \\
\end{bmatrix}
\]

\[
1 \leq k, \ell \leq b
\]

where \( b = b_j(X_\Gamma) \), the \( \gamma_k \in H_j(X_\Gamma) \) project to a basis for the free part, and the \( \omega_\ell \)'s are an orthonormal basis for the space of harmonic \( j \)-forms on \( X_\Gamma \).
Regulators

- The ‘degree $j$ regulator’ is the determinant

$$R_j(X_\Gamma) = \det \left( \int_{\gamma_k} \omega_\ell \right)_{1 \leq k, \ell \leq b}$$

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- We have:
  
  $$|R_0(X_\Gamma)| = 1/\sqrt{\text{vol}(X_\Gamma)}, \quad |R_n(X_\Gamma)| = \sqrt{\text{vol}(X_\Gamma)},$$

  and by Poincaré duality, we have

  $$|R_j(X_\Gamma)R_{n-j}(X_\Gamma)| = 1.$$
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- **Conjecture.** Fix $S$ and $j$. The growth of the degree $j$ regulators of congruence arithmetic $S$-manifolds is given by
  \[ \log |R_j(X_\Gamma)| = o(\text{vol}(X_\Gamma)). \]
– **Conjecture.**– For congruence arithmetic $S$-manifolds, we have:

\[ \log |R_j(X_\Gamma)| = o(\text{vol}(X_\Gamma)) \].

Conjecture false for non-arithmetic manifolds: Brock and Dunfield construct closed hyperbolic 3-manifolds $M$, of arbitrary large volume, s.t. $b_1(M) = 1$, and

\[ \limsup \frac{\log |R_2(M)|}{\text{vol}(M)} > 0. \]
On the conjecture

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  \]

- B.-Şengün-Venkatesh: verify Conjecture when $X_\Gamma$ is a congruence cover of a Bianchi manifold with 1-dimensional cuspidal cohomology associated to a non-CM elliptic curve. The proof relates the complexity of the $H_2$-cycle to the height of the associated elliptic curve.
Thanks to:

Haglund, Leys, Li, Millson, Moeglin, Şengün, Venkatesh, Wise