On Explicit Aspect of Pluricanonical Maps of Projective Varieties

Jungkai A. Chen & Meng Chen

National Taiwan University & Fudan University

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Outline

I–1.1. Birational Geometry

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- pair (X, Δ), or generalized pair $(X, B + M)$
- pluricanonical divisor, mK_X
- twisted canonical divisor, $K_X + P$ for some $P \in \mathrm{Pic}^0$.

I–1.2. Pluricanonical Maps and Iitaka fibration

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$$\varphi_m : X \dashrightarrow \mathbb{P}^N.$$

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φ_m is called the *m*-th canonical map.
There exist $d(X)$ and $r(X)$ such that
φ_m is stabilized (birationally) for $m \geq r(X)$ and divisible by $d(X)$.

I–1.3. Fundamental Questions in Explicit Geometry

- Canonical stability index.

For any integer $n \geq 3$, find a practical integer r_n so that, for all nonsingular projective n-folds of general type, φ_m is birational onto its image for all $m \geq r_n$.

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- **Iitaka fibrations.**

For any integers $n \geq 3$ and $n > \kappa \geq 0$, find integers $M_{n,\kappa}$ and $d_{n,\kappa}$ such that, for all nonsingular projective n-folds with Kodaira dimension κ, the m-th canonical map φ_m defines an Iitaka fibration for all $m \geq M_{n,\kappa}$ and divisible by $d_{n,\kappa}$.

I–1.4. Fundamental Questions in Explicit Geometry

- Anti-pluricanonical birationality.

For any integer $n \geq 3$, find an integer m_n so that, for all canonical (terminal) weak $\mathbb{Q}$-Fano n-folds (i.e. $-K$ being $\mathbb{Q}$-Cartier, nef and big), φ_{-m} is birational onto its image for all $m \geq m_n$.

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birationally invariant, when $\kappa \geq 0$ and X has canonical singularities;

NOT birationally invariant, when $\kappa = -\infty$.

I–1.5. Known Existence Results

- [Hacon-McKernan '06, Takayama '06, Tsuji '06]

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If $\kappa(X) = 1$, then $M_{n,1}$ and $d_{n,1}$ exist.

- [Viehweg-Zhang '07]

If $\kappa(X) = 2$, then $M_{n,2}$ and $d_{n,2}$ exist.

I–1.6. Known Existence Results

- [Birkar-Zhang '16]

There exists a uniform number $M(n, b_F, \beta_{\tilde{F}})$ so that φ_m gives an Iitaka fibration for all $m \geq M(n, b_F, \beta_{\tilde{F}})$ and divisible.

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Let F be the general fiber of Iitaka fibration. The number b_F, called the *index of fiber*, is the smallest positive integer so that $|bK_F| \neq \emptyset$.
One has a covering $\tilde{F} \to F$ by $|mK_F|$. Then $\beta_{\tilde{F}}$, called the *middle Betti number*, is defined as the $(n-\kappa)$-th Betti number of the $n-\kappa$ dimensional variety $\tilde{F}$.

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- [Kollár–Miyaoka–Mori–Takagi, '00] m_3 exists.
- [Birkar, '16] for $n \geq 4$ there is a constant m_n depending only on n such that φ_{-m} is birational for all $m \geq m_n$.

I–1.8. Explicit Geometry in Higher Dimensional

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I–2.1. Pluricanonical Maps on Irregular Varieties

A variety is said to be *irregular* if $h^1(\mathcal{O}_X) > 0$.
In other words, there exists non-trivial Albanese map

$$a_X \colon X \to \mathrm{Alb}(X).$$

Let $\mathfrak{a}(X) = \dim(a_X(X))$.

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Theorem

[Chen-Hacon '07]
Let X be a variety of $\kappa(X) = \mathfrak{a}(X) = \dim X$.
Suppose that $\chi(X, \mathcal{O}(K_X)) > 0$, then $(a_X)_*\mathcal{O}(K_X)$ is a M-regular sheaf.

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Moreover, $|3K_X|$ is birational.

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Some More Recent Results:

- [Z. Jiang, Lahos, Tirabashi, '14]
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- [J. Chen, M. Chen, Z. Jiang, '16]
  Let X be an irregular theefold of general type. Then $|6K|$ is birational.
  The most difficult case is $X \to \mathrm{Alb}(X)$ is a morphism to an elliptic curve fibered by surface of $(1, 2)$-type.

From now on, we will concentrate on threefolds

I–3.1. Baskets of Terminal Orbifold Points

- A terminal orbifold point of type $\frac{1}{r}(1, -1, b)$ will be denoted as (b, r) with $b \leq r/2$.

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- A terminal orbifold point of type $\frac{1}{r}(1,-1,b)$ will be denoted as (b,r) with $b \le r/2$.
- A basket, which is a collection of terminal orbifold points, is written as $\mathcal{B}=\{n_i \times (b_i,r_i)\}$ where n_i denotes the multiplicities.

I–3.2. Plurigenus Formula

- Reid's Riemann-Roch formula for singular threefolds:

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}(D.c_2(X)) + \sum_{P\in B(X)}\left(-i_P\cdot\frac{r_P^2-1}{12r_P} + \sum_{j=1}^{i_P-1}\frac{\overline{jb_P}(r_P-\overline{jb_P})}{2r_P}\right),$$

where $B(X)=\{(b_P,r_P)\}$ is the basket data of X and i_P is the local index of D such that $\mathcal{O}_X(D)\cong\mathcal{O}_X(i_PK_X)$ near P.

I–3.3. Plurigenus Formula

- Take $D = K_X$, then one gets

$$(K_X.c_2(X)) = -24\chi(\mathcal{O}_X) + \sum_{P \in B_X} \left(r_P - \frac{1}{r_P} \right).$$

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• Taking $D = mK_X$. One gets the following plurigenus formula (due to Reid):

$$\chi_m = \frac{1}{12}m(m-1)(2m-1)K^3 + (1-2m)\chi + l(m), \qquad (1)$$

where $\chi = \chi(\mathcal{O}_X)$, $K^3 = K_X^3$, $\chi_m = \chi(\mathcal{O}_X(mK_X))$ and

$$l(m) = \sum_{P\in B_X}\sum_{j=1}^{m-1}\frac{\overline{jb_P}(r_P - \overline{jb_P})}{2r_P}. \qquad (2)$$

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- The triple (B_X, χ_2, χ) determines $K^3(\mathbb{B})$.

I–3.5. "Packings" between Baskets

- Given a basket

$$B = \{(b_1, r_1), (b_2, r_2), \ldots, (b_k, r_k)\},$$

we call the basket

$$B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \ldots, (b_k, r_k)\}$$

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- If $b_1 r_2 - b_2 r_1 = 1$, then we call $B \succcurlyeq B'$ a prime packing.

I–3.6. The canonical Sequence of a Basket

• The packing of baskets naturally induces the packing of weighted baskets, namely we define

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• The basket $B^{(0)}$, called the initial basket, consists of orbifold points of the form $(1, r_i)$.

I–3.7. Main Properties of the Packing

Proposition

Assume $\mathbb{B} \succcurlyeq \mathbb{B}'$. *Then*

- $P_m(\mathbb{B}) \geq P_m(\mathbb{B}')$ *for all* $m \geq 2$;

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Assume $\mathbb{B} \succcurlyeq \mathbb{B}'$. *Then*

- $P_m(\mathbb{B}) \geq P_m(\mathbb{B}')$ *for all* $m \geq 2$;
- $K^3(\mathbb{B}) \geq K^3(\mathbb{B}')$.

I–3.8. The Key Inequality

• the canonical sequence provide many new inequalities among the Euler characteristic
Of which the most interesting one is:

$$2\chi_5+3\chi_6+\chi_8+\chi_{10}+\chi_{12} \geq \chi+10\chi_2+4\chi_3+\chi_7+\chi_{11}+\chi_{13}+R, \quad (3)$$

where R is certain non-negetative combination of all initial baskets with higher indices.

I–3.9. Application of Basket Theory

General applications:

- K_X (resp. $-K_X$) is nef and big, then $\chi_m = P_m$ for $m \geq 2$ (resp. $m \leq -1$)

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  One can study the geometry of X by using the map φ_m.
- The set $\{\mathbb{B} | \chi_m(\mathbb{B}) < 2, m \leq m_0\}$ is finite and can be classified.
- For any given weighted basket, one can find $m'(\mathbb{B})$ such that $p_{m'} = \chi_{m'(\mathbb{B})} \geq 2$.

I–3.10. Weighted Complete Intersection

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- We also prove that there is no more example if degree > 100, by using theory of baskets.
- —The end of part one—

II–1.1. The case $r_X = 1$ (—"Gorenstein minimal")

- Let X be a minimal 3-fold of general type. Recall the *canonical stability index*

$$r_{\mathrm{s}}(X) = \min\{t \in \mathbb{Z}_{>0} \mid \varphi_{m,X} \text{ is birational for all } m \geq t\}.$$

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• Finally proved by Chen-Chen-Zhang (2007):

Theorem

Let X be a minimal projective 3-fold of general type with $r_X = 1$. Then $\varphi_{m,X}$ is a biratioal morphism for every integer $m \geq 5$.

II–1.2. Kollár's Method

- Kollár's result in 1986:

Theorem

Let V be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then φ_{11k+5} is birational.

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- Kollár's method: taking a sub-pencil $\Lambda \subset |kK_V|$, one gets a surjective morphism $f\colon V \longrightarrow \Gamma \cong \mathbb{P}^1$. One has the inclusion $\mathcal{O}(1) \hookrightarrow f_*\omega_V^k$ and then, for any $p \geq 5$,

$$f_*\omega_{V/\Gamma}^p \otimes \mathcal{O}(1) \hookrightarrow f_*\omega_V^{(2p+1)k+p}.$$

Since the 5-canonical map of the general fiber is birational and by the semi-positivity of $f_*\omega_{V/\Gamma}^p$, one sees that φ_{11k+5} is birational by simply taking $p = 5$.

II–1.3. Improved form of Kollár's Theorem

- Proved by M. Chen in 2004:

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Let V be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then φ_m is birational for all $m \geq 5k + 6$.

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- Kollár's method + the geometry of linear systems

II–1.4. The case $r_X \geq 2$

- Suppose $\chi(\mathcal{O}_X) < 0$. Reid's Riemann-Roch formula implies $P_2(X) \geq 4$. Hence the question is solvable by Kollár's theorem.

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• Suppose that $P_m \geq 2$ for some $m \leq 12$, one applies Kollár's method as well.

• The remain situation: $\chi(\mathcal{O}_X) \geq 0$ and $P_k(X) \leq 1$ for all $2 \leq k \leq 12$. Key Inequality reads:

$$2P_5+3P_6+P_8+P_{10}+P_{12} \geq \chi(\mathcal{O}_X)+10P_2+4P_3+P_7+P_{11}+P_{13}, \tag{4}$$

which directly implies that $\chi(\mathcal{O}_X) \leq 8$, $P_{13} \leq 7$.

II–1.5. Boundedness Results

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- Chen-Chen 2010-2015:

Theorem

Let X be a minimal projective 3-fold of general type. Then

(1) $K_X^3 \geq \frac{1}{1680}$;

(2) $\varphi_{m,X}$ *is birational for all* $m \geq 61$;

(3) $P_{12} \geq 1$ *and* $P_{24} \geq 2$.

(4) $K_X^3 \geq \frac{1}{420}$ *(optimal) if* $\chi(\mathcal{O}_X) \leq 1$.

II–1.6. Explicit Classifications

- Define the <u>*pluricanonical section index*</u> $\delta(X)$ to be the minimal integer so that $P_\delta \geq 2$.

Theorem

Let X be a minimal projective 3-fold of general type. Then

(1) $\delta(X) \leq 18$;

(2) $\delta(X) = 18$ *if and only if* $\mathbb{B}(X) = \{B_{2a}, 0, 2\}$;

(3) $\delta(X) \neq 16,\ 17$;

(4) $\delta(X) = 15$ *if and only if* $\mathbb{B}(X)$ *belongs to one of the types in [CC3,* Table F–1*]*;

(5) $\delta(X) = 14$ *if and only if* $\mathbb{B}(X)$ *belongs to one of the types in [CC3,* Table F–2*]*;

(6) $\delta(X) = 13$ *if and only if* $\mathbb{B}(X) = \{B_{41}, 0, 2\}$.

II–1.7. The Up-to-date Result!

- Recently M. Chen showed $r_3 \leq 57$ on the basis of above classifications. Therefore, $\underline{27 \leq r_3 \leq 57}$.

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Theorem

Let X be a minimal projective 3-fold of general type with $p_g(X) \geq 2$. Then

(1) $K_X^3 \geq \frac{1}{3}$;

(2) $\varphi_{8,X}$ *is birational onto its image.*

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- For 3-folds with $\delta = 2$, Chen-Chen proved that $r_s(X) \leq 11$ (optimal).

II–2.1. $\mathbb{Q}$-Fano 3-folds

• A normal projective 3-fold X is called a weak $\mathbb{Q}$-Fano 3-fold (resp. $\mathbb{Q}$-*Fano 3-fold*) if the anti-canonical divisor $-K_X$ is nef and big (resp. ample). A *canonical* (resp. *terminal)* weak $\mathbb{Q}$-Fano 3-fold is a weak $\mathbb{Q}$-Fano 3-fold with at worst canonical (resp. terminal) singularities.

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- Take the weighted basket

$$\mathbb{B}(X) = \{B_X, P_{-1}, \chi(\mathcal{O}_X)\}.$$

By the duality and the vanishing of higher cohomology, we always have $\chi_m = -P_{-(m-1)}$ for all $m \geq 2$. Hence the basket theory has a parallel version in Fano case.

II–2.2. Lower Bound of the Anti-canonical Volume

• In 2008, Chen-Chen applied the basket theory to prove the following theorem:

Theorem

Let X be a terminal (or canonical) weak $\mathbb{Q}$-Fano 3-fold. Then

(1) $P_{-4} > 0$ *with possibly one exception of a basket of singularities;*

(2) $P_{-6} > 0$ *and* $P_{-8} > 1$*;*

(3) $-K_X^3 \geq \frac{1}{330}$. *Furthermore* $-K_X^3 = -\frac{1}{330}$ *if and only if the basket of singularities is* $\{(1,2),(2,5),(1,3),(2,11)\}$.

The above theorem is optimal according to Fletcher: $X_{66} \subset \mathbb{P}(1,5,6,22,33)$.

II–2.3. The Anti-pluricanonical Birationality

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- Chen-Jiang proved in 2016:

Theorem

Let X be a terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then $\varphi_{-m,X}$ is birational for all $m \geq 39$.

Theorem

Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $\varphi_{-m,X}$ is birational for all $m \geq 97$.

II–2.4. The Anti-pluricanonical Birationality

- "$m_3 \leq 97$" is far from being optimal!

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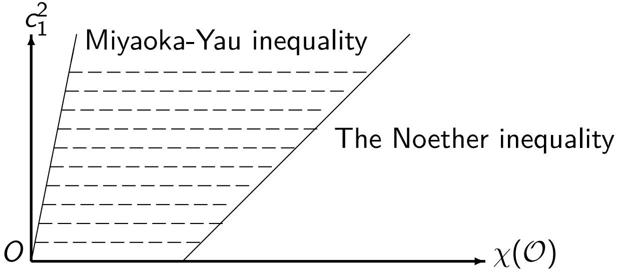
- "$m_3 \leq 97$" is far from being optimal!
- Chen-Jiang proved in 2017:

Theorem

Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then, for any K-Mori fiber space Y of V, $\varphi_{-m,Y}$ is birational for all $m \geq 52$.

II–3.1. Recall–The Surface Geography

- General strategy of the geography



II–3.2. The Noether Inequality

• There is no effective 3-dimensional analogy of Miyaoka-Yau inequality "$K_S^2 \leq 9\chi(\mathcal{O}_S)$", since $-\infty < \chi(\mathcal{O}_X) < +\infty$.

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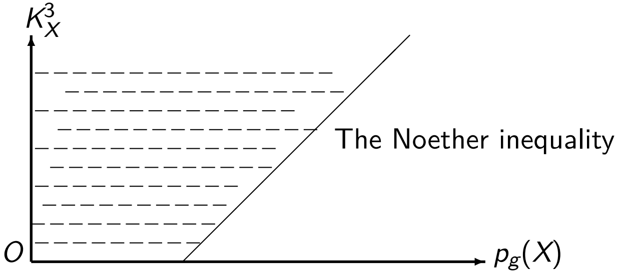
- Seek for the Noether inequality!



II–3.3. History of 3-Dimensional Noether Inequality

- X minimal, the Cartier index $r_X \geq 1$.

$$X \text{ is Gorenstein} \iff r_X = 1$$

$$\{\text{smooth minimals}\} \subset \{\text{Grenstein minimals}\} \subset \{\text{General minimals}\}$$

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- X minimal, the Cartier index $r_X \geq 1$.

$$X \text{ is Gorenstein } \iff r_X = 1$$

{smooth minimals} $\subset$ {Grenstein minimals} $\subset$ {General minimals}

- The possible Noether type inequality is of the form:

$$\underline{K_X^3 \geq a\, p_g(X) - b}$$

$a, b \in \mathbb{Q}_{>0}$.

II–3.4. The Noether Inequality for Gorenstein Minimal 3-folds

- Kobayashi (1992): an infinite series of examples of canonically polarized 3-folds satisfying $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$.

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- M. Chen (2004): $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for canonically polarized 3-folds.

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- Catanese-Chen-Zhang (2006): $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for smooth minimal 3-folds of general type.

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- Catanese-Chen-Zhang (2006): $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for smooth minimal 3-folds of general type.
- Chen-Chen (2015): $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for Gorenstein minimal 3-folds of general type.

II–3.5. Arbitrary minimal 3-folds

- May always assume $p_g(X) \geq 2$, since $K_X^3 > 0$. So $\varphi_{|K_X|}$ is non-trivial.

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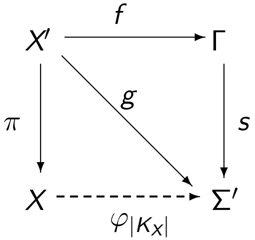
- May always assume $p_g(X) \geq 2$, since $K_X^3 > 0$. So $\varphi_{|K_X|}$ is non-trivial.
- Set up for $\varphi_1 = \varphi_{|K_X|}$. Set $d_X = \dim(\Gamma)$.



II–3.6. The main statement

Theorem

Let X be a minimal projective 3-fold of general type. Assume that one of the following holds:

- $d_X \geq 2$; *or*
- $d_X = 1$ *and* $|K_X|$ *is not composed with a rational pencil of* $(1,2)$*-surfaces; or*
- $d_X = 1$, $|K_X|$ *is composed with a rational pencil of* $(1,2)$*-surfaces, and either* $p_g(X) \leq 4$ *or* $p_g(X) \geq \frac{2}{glct(1,2)} + 1$.

Then the inequality

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$$

holds.

II–3.7. The Noether Inequality for Algebraic 3-folds

- János Kollár: $\mathrm{glct}(1,2) \geq \frac{1}{10}$ (optimal).

II–3.7. The Noether Inequality for Algebraic 3-folds

- János Kollár: $\mathrm{glct}(1,2) \geq \frac{1}{10}$ (optimal).
- Chen-Chen-Jiang (2018) proved the following:

Theorem

Let X be a minimal projective 3-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 21$. Then the inequality holds:

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}.$$

Corollary

The inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ holds except for finite number of families of 3-folds of general type.

II–3.8. Conjecture A

- The 3D-Noether Inequality

Conjecture

The inequality $K^3 \geq \frac{4}{3}p_g - \frac{10}{3}$ holds for all minimal 3-folds of general type with $5 \leq p_g \leq 20$.

II–3.8. Conjecture A

- The 3D-Noether Inequality

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The inequality $K^3 \geq \frac{4}{3}p_g - \frac{10}{3}$ holds for all minimal 3-folds of general type with $5 \leq p_g \leq 20$.

- Projective varieties with very large canonical volumes. For $n \geq 2$, recall:

$$r_n = \max\{r_s(X) \mid X \text{ is a n-fold of general type}\};$$

$$r_n^+ = \max\{r_s(X) \mid X \text{ is a n-fold of general type with } p_g > 0\};$$

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$$r_n^+ = \max\{r_s(X) \,|X \text{ is a n-fold of general type with } p_g > 0\};$$

- By definition, one has $r_n^+ \leq r_n$.

II–3.9. Conjecture B and Conjecture C

- Conjecture B.

Conjecture

There exists a function $K(n)$ such that $r_s(X) \leq r_{n-1}$ holds for any $n \geq 5$ and for any minimal projective n-fold X with $K_X^n > K(n)$.

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- Conjecture C.

Conjecture

There exists a function $L(n)$ such that $r_s(X) \leq r_{n-1}^+$ holds for any $n \geq 6$ and for any minimal projective n-fold X of general type with $p_g > L(n)$.

Thank you very much!