On Explicit Aspect of Pluricanonical Maps of Projective Varieties

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I-1. Introduction

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Birational Classification:

to classify varieties up to birational equivalence.
I–1.1. Birational Geometry

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• **Birational Geometry:**
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- **pair** \((X, \Delta)\), or **generalized pair** \((X, B + M)\)

- **pluricanonical divisor**, \(mK_X\)

- **twisted canonical divisor**, \(K_X + P\) for some \(P \in \text{Pic}^0\).
Given a complex projective variety $X$ and let $K_X$ be the canonical divisor. Suppose that $H^0(X, mK_X) \neq 0$, then we have a natural map

$$\varphi_m : X \to \mathbb{P}^N.$$
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Given a complex projective variety $X$ and let $K_X$ be the canonical divisor. Suppose that $H^0(X, mK_X) \neq 0$, then we have a natural map

$$\varphi_m : X \dashrightarrow \mathbb{P}^N.$$ 

$\varphi_m$ is called the $m$-th canonical map. There exist $d(X)$ and $r(X)$ such that $\varphi_m$ is stabilized (birationally) for $m \geq r(X)$ and divisible by $d(X)$. 

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• **Canonical stability index.**

For any integer $n \geq 3$, find a practical integer $r_n$ so that, for all nonsingular projective $n$-folds of general type, $\varphi_m$ is birational onto its image for all $m \geq r_n$. 
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• **Iitaka fibrations.**
For any integers $n \geq 3$ and $n > \kappa \geq 0$, find integers $M_{n,\kappa}$ and $d_{n,\kappa}$ such that, for all nonsingular projective $n$-folds with Kodaira dimension $\kappa$, the $m$-th canonical map $\varphi_m$ defines an Iitaka fibration for all $m \geq M_{n,\kappa}$ and divisible by $d_{n,\kappa}$.
• Anti-pluricanonical birationality.
For any integer $n \geq 3$, find an integer $m_n$ so that, for all canonical (terminal) weak $\mathbb{Q}$-Fano $n$-folds (i.e. $-K$ being $\mathbb{Q}$-Cartier, nef and big), $\varphi_{-m}$ is birational onto its image for all $m \geq m_n$. 
I-1. Introduction

I–1.4. Fundamental Questions in Explicit Geometry

• Anti-pluricanonical birationality.

For any integer $n \geq 3$, find an integer $m_n$ so that, for all canonical (terminal) weak $\mathbb{Q}$-Fano $n$-folds (i.e. $-K$ being $\mathbb{Q}$-Cartier, nef and big), $\varphi_{-m}$ is birational onto its image for all $m \geq m_n$.

• The behavior of $\varphi_{m,X}$ is birationally invariant, when $\kappa \geq 0$ and $X$ has canonical singularities;
I–1.4. Fundamental Questions in Explicit Geometry

- **Anti-pluricanonical birationality.**
  For any integer \( n \geq 3 \), find an integer \( m_n \) so that, for all canonical (terminal) weak \( \mathbb{Q} \)-Fano \( n \)-folds (\( -K \) being \( \mathbb{Q} \)-Cartier, nef and big), \( \varphi_{-m} \) is birational onto its image for all \( m \geq m_n \).

- The behavior of \( \varphi_{m,X} \) is **birationally invariant**, when \( \kappa \geq 0 \) and \( X \) has canonical singularities;
  **NOT birationally invariant**, when \( \kappa = -\infty \).

I–1.5. Known Existence Results

- [Hacon–McKernan ’06, Takayama ’06, Tsuji ’06] If $\kappa(X) = \dim X$, then $r_n$ exists.
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- [Hacon-McKernan ’06, Takayama ’06, Tsuji ’06] If \( \kappa(X) = \dim X \), then \( r_n \) exists.
- [Fujino-Mori ’00] If \( \kappa(X) = 1 \), then \( M_{n,1} \) and \( d_{n,1} \) exist.
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  If $\kappa(X) = \dim X$, then $r_n$ exists.

- [Fujino-Mori ’00]
  If $\kappa(X) = 1$, then $M_{n,1}$ and $d_{n,1}$ exist.

- [Viehweg-Zhang ’07]
  If $\kappa(X) = 2$, then $M_{n,2}$ and $d_{n,2}$ exist.
● [Birkar-Zhang '16]
There exists a uniform number $M(n, b_F, \beta_{\tilde{F}})$ so that $\varphi_m$ gives an Iitaka fibration for all $m \geq M(n, b_F, \beta_{\tilde{F}})$ and divisible.
I-1. Introduction

I–1.6. Known Existence Results

- [Birkar-Zhang ’16]
  There exists a uniform number $M(n, b_F, \beta_{\tilde{F}})$ so that $\varphi_m$ gives an Iitaka fibration for all $m \geq M(n, b_F, \beta_{\tilde{F}})$ and divisible.
  Let $F$ be the general fiber of Iitaka fibration. The number $b_F$, called the *index of fiber*, is the smallest positive integer so that $|bK_F| \neq \emptyset$. 
I–1.6. Known Existence Results

• [Birkar-Zhang '16]
  There exists a uniform number $M(n, b_F, \beta_{\tilde{F}})$ so that $\varphi_m$ gives an Iitaka fibration for all $m \geq M(n, b_F, \beta_{\tilde{F}})$ and divisible. Let $F$ be the general fiber of Iitaka fibration. The number $b_F$, called the index of fiber, is the smallest positive integer so that $|bK_F| \neq \emptyset$.
  One has a covering $\tilde{F} \to F$ by $|mK_{\tilde{F}}|$. Then $\beta_{\tilde{F}}$, called the middle Betti number, is defined as the $(n - \kappa)$-th Betti number of the $n - \kappa$ dimensional variety $\tilde{F}$.
I–1.7. $\kappa = -\infty$

- [Kawamata, ’89] for weak $\mathbb{Q}$-Fano threefolds, the boundedness was proved under the condition that the Picard number $\rho = 1$
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- [Kollár–Miyaoka–Mori–Takagi, ’00] \( m_3 \) exists.
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- [Kawamata, ’89] for weak $\mathbb{Q}$-Fano threefolds, the boundedness was proved under the condition that the Picard number $\rho = 1$
- [Kollár–Miyaoka–Mori–Takagi, ’00] $m_3$ exists.
- [Birkar, ’16] for $n \geq 4$ there is a constant $m_n$ depending only on $n$ such that $\varphi_{-m}$ is birational for all $m \geq m_n$. 
Can we find those above mentioned bound explicitly?
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At least in dimension three?
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Or in some special interesting category?
I-1.8. Explicit Geometry in Higher Dimensional

- Can we find those above mentioned bound explicitly?
- At least in dimension three?
- Or in some special interesting category?
A variety is said to be *irregular* if $h^1(\mathcal{O}_X) > 0$. In other words, there exists non-trivial *Albanese map*

$$a_X : X \rightarrow \text{Alb}(X).$$

Let $a(X) = \dim(a_X(X))$. 

*Theorem* [Chen-Hacon '07]

Let $X$ be a variety of $(X) = a(X) = \dim X$. Suppose that $(X; \mathcal{O}_K^\times X) > 0$, then $(a_X)$ is a $M$-regular sheaf. Moreover, $j_3 K^\times X$ is birational.

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I–2. Irregular Varieties

I–2.1. Pluricanonical Maps on Irregular Varieties

A variety is said to be irregular if $h^1(\mathcal{O}_X) > 0$. In other words, there exists non-trivial Albanese map

$$a_X : X \to \text{Alb}(X).$$

Let $a(X) = \dim(a_X(X))$.

**Theorem**

[Chen-Hacon '07]

Let $X$ be a variety of $\kappa(X) = a(X) = \dim X$. Suppose that $\chi(X, \mathcal{O}(K_X)) > 0$, then $(a_X)_* \mathcal{O}(K_X)$ is a $M$-regular sheaf.
A variety is said to be *irregular* if $h^1(\mathcal{O}_X) > 0$. In other words, there exists non-trivial *Albanese map* 

$$a_X : X \to \text{Alb}(X).$$ 

Let $\alpha(X) = \dim(a_X(X))$.

**Theorem**

[Chen-Hacon '07]

Let $X$ be a variety of $\kappa(X) = \alpha(X) = \dim X$. Suppose that $\chi(X, \mathcal{O}(K_X)) > 0$, then $(a_X)_* \mathcal{O}(K_X)$ is a $M$-regular sheaf. Moreover, $|3K_X|$ is birational.
Some More Recent Results:

- [Z. Jiang, Lahos, Tirabashi, ’14]
  \( \kappa(X) = \alpha(X) = \dim X \). Then \( |3K| \) is birational.
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- [Z. Jiang, H. Sun, ’14]
  \( \kappa(X) = \dim X, \ a(X) = \dim X - 1 \). Then \( |4K| \) is birational.
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  $\kappa(X) = \dim X$, $\alpha(X) = \dim X - 1$. Then $|4K|$ is birational.

- [J. Chen, M. Chen, Z. Jiang, '16]
  Let $X$ be an irregular threefold of general type. Then $|6K|$ is birational.
  The most difficult case is $X \to \text{Alb}(X)$ is a morphism to an elliptic curve fibered by surface of $(1, 2)$-type.
From now on, we will concentrate on threefolds
I–3.1. Baskets of Terminal Orbifold Points

• A terminal orbifold point of type $\frac{1}{r}(1, -1, b)$ will be denoted as $(b, r)$ with $b \leq r/2$. 
I–3. Baskets and Weighted Complete Intersections (WCI)

I–3.1. Baskets of Terminal Orbifold Points

- A terminal orbifold point of type $\frac{1}{r}(1, -1, b)$ will be denoted as $(b, r)$ with $b \leq r/2$.
- A basket, which is a collection of terminal orbifold points, is written as $\mathcal{B} = \{ n_i \times (b_i, r_i) \}$ where $n_i$ denotes the multiplicities.
I–3.2. Plurigenus Formula

- **Reid’s** Riemann-Roch formula for singular threefolds:

\[
\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} (D.c_2(X)) \\
+ \sum_{P \in B(X)} \left( -i_P \cdot \frac{r_P^2 - 1}{12r_P} + \sum_{j=1}^{i_P - 1} \frac{j b_P(r_P - j b_P)}{2r_P} \right),
\]

where \( B(X) = \{(b_P, r_P)\} \) is the basket data of \( X \) and \( i_P \) is the local index of \( D \) such that \( \mathcal{O}_X(D) \cong \mathcal{O}_X(i_P K_X) \) near \( P \).
I–3.3. Plurigenus Formula

• Take $D = K_X$, then one gets

$$(K_X . c_2(X)) = -24 \chi(O_X) + \sum_{P \in B_X} \left( r_P - \frac{1}{r_P} \right).$$
I–3.3. Plurigenus Formula

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$$(K_X.c_2(X)) = -24\chi(O_X) + \sum_{P \in B_X} \left( r_P - \frac{1}{r_P} \right).$$

- Taking $D = mK_X$. One gets the following plurigenus formula (due to Reid):

$$\chi_m = \frac{1}{12} m(m - 1)(2m - 1)K^3 + (1 - 2m)\chi + l(m), \quad (1)$$

where $\chi = \chi(O_X)$, $K^3 = K_X^3$, $\chi_m = \chi(O_X(mK_X))$ and

$$l(m) = \sum_{P \in B_X} \sum_{j=1}^{m-1} \frac{j b_P(r_P - \overline{j b_P})}{2r_P}. \quad (2)$$
I–3.4. Weighted Baskets

• We call the triple $\mathbb{B} = \{B, \chi_2, \chi\}$ a weighted basket.
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The triple $(B_X, \chi_2, \chi)$ determines $\chi_m$ for all $m \geq 3$. 
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- The triple $(B_X, \chi_2, \chi)$ determines $\chi_m$ for all $m \geq 3$.
- The triple $(B_X, \chi_2, \chi)$ determines $K^3(\mathbb{B})$. 
Given a basket

\[ B = \{(b_1, r_1), (b_2, r_2), \ldots, (b_k, r_k)\}, \]

we call the basket

\[ B' = \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \ldots, (b_k, r_k)\} \]

a packing of \( B \), written as \( B \succ B' \).
I-3. Baskets and Weighted Complete Intersections (WCI)

I–3.5. “Packings” between Baskets

- Given a basket

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a packing of \( B \), written as \( B \succ B' \).

- If \( b_1 r_2 - b_2 r_1 = 1 \), then we call \( B \succ B' \) a prime packing.
I–3.6. The canonical Sequence of a Basket

- The packing of baskets naturally induces the packing of weighted baskets, namely we define

\[ \mathcal{B} = \{ B, \chi_2, \chi \} \succ \{ B', \chi_2, \chi \} = \mathcal{B}' \]

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- The “canonical sequence of a basket”:

\[ B^{(0)}(B) \succeq B^{(5)}(B) \succeq ... \succeq B^{(n)}(B) \succeq ... \succeq B. \]
I-3. Baskets and Weighted Complete Intersections (WCI)

I–3.6. The canonical Sequence of a Basket

• The packing of baskets naturally induces the packing of weighted baskets, namely we define

\[ \mathcal{B} = \{ B, \chi_2, \chi \} \cong \{ B', \chi_2, \chi \} = \mathcal{B}' \]

if \( B \supseteq B' \).

• The “canonical sequence of a basket”:

\[ B^{(0)}(B) \cong B^{(5)}(B) \cong \ldots \cong B^{(n)}(B) \cong \ldots \cong B. \]

• The basket \( B^{(0)} \), called the initial basket, consists of orbifold points of the form \( (1, r_i) \).
Proposition

Assume \( B \succeq B' \). Then

\[ P_m(B) \geq P_m(B') \text{ for all } m \geq 2; \]
I–3.7. Main Properties of the Packing

Proposition

Assume \( B \succcurlyeq B' \). Then

- \( P_m(B) \geq P_m(B') \) for all \( m \geq 2 \);
- \( K^3(B) \geq K^3(B') \).
I-3.8. The Key Inequality

- The canonical sequence provide many new inequalities among the Euler characteristic.
  Of which the most interesting one is:

\[ 2\chi_5 + 3\chi_6 + \chi_8 + \chi_{10} + \chi_{12} \geq \chi + 10\chi_2 + 4\chi_3 + \chi_7 + \chi_{11} + \chi_{13} + R, \quad (3) \]

where \( R \) is certain non-negative combination of all initial baskets with higher indices.
General applications:

- $K_X$ (resp. $-K_X$) is nef and big, then $\chi_m = P_m$ for $m \geq 2$
  (resp. $m \leq -1$)
I–3.9. Application of Basket Theory

General applications:

- $K_X$ (resp. $-K_X$) is nef and big, then $\chi_m = P_m$ for $m \geq 2$
  (resp. $m \leq -1$)
- Suppose that $\chi_m \geq 2$ for some $m \leq m_0$, then there exists a non-trivial $\varphi_m$.
  One can study the geometry of $X$ by using the map $\varphi_m$. 
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- One can study the geometry of $X$ by using the map $\varphi_m$.
- The set $\{\mathcal{B} | \chi_m(\mathcal{B}) < 2, m \leq m_0\}$ is finite and can be classified.
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General applications:

- $K_X$ (resp. $-K_X$) is nef and big, then $\chi_m = P_m$ for $m \geq 2$ (resp. $m \leq -1$)
- Suppose that $\chi_m \geq 2$ for some $m \leq m_0$, then there exists a non-trivial $\varphi_m$.
  One can study the geometry of $X$ by using the map $\varphi_m$.
- The set $\{B | \chi_m(B) < 2, m \leq m_0\}$ is finite and can be classified.
- For any given weighted basket, one can find $m'(B)$ such that $p_{m'} = \chi_{m'}(B) \geq 2$. 
Fletcher gave lists of canonically polarized (resp. anti-canonically polarized) weighted complete intersections threefolds of codimension $\leq 5$ (resp. 3) of degree $\leq 100$. 

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Fletcher gave lists of canonically polarized (resp. anti-canonically polarized) weighted complete intersections threefolds of codimension $\leq 5$ (resp. $3$) of degree $\leq 100$.

We prove there is no more examples if codimension $> 5$ (resp. $> 3$).
I–3.10. Weighted Complete Intersection

Fletcher gave lists of canonically polarized (resp. anti-canonically polarized) weighted complete intersections threefolds of codimension $\leq 5$ (resp. 3) of degree $\leq 100$.

We prove there is no more examples if codimension $> 5$ (resp. $> 3$).

We also prove that there is no more example if degree $> 100$, by using theory of baskets.
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—The end of part one—
II–1. The case $r_X = 1$ (—“Gorenstein minimal”)

- Let $X$ be a minimal 3-fold of general type. Recall the **canonical stability index**

$$r_s(X) = \min\{ t \in \mathbb{Z}_{>0} \mid \varphi_{m,X} \text{ is birational for all } m \geq t \}.$$
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$$r_s(X) = \min\{ t \in \mathbb{Z}_{>0} \mid \varphi_{m,X} \text{ is birational for all } m \geq t \}.$$

- When $X$ is smooth and minimal, Wilson proved $r_s(X) \leq 25$. Then improved, chronologically, by Benveniste ($\leq 9$), Matsuki ($\leq 7$) and M. Chen ($\leq 6$).
II–1. Explicit birational geometry for 3-folds of general type

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  **canonical stability index**

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- Finally proved by **Chen-Chen-Zhang (2007):**

**Theorem**

*Let $X$ be a minimal projective 3-fold of general type with $r_X = 1$. Then $\varphi_{m,X}$ is a birational morphism for every integer $m \geq 5$.*
II–1.2. Kollár’s Method

- **Kollár’s result in 1986:**

**Theorem**

Let $V$ be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then $\varphi_{11k+5}$ is birational.
II–1.2. Kollár’s Method

- **Kollár’s result in 1986:**

Theorem

Let $V$ be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then $\varphi_{11k+5}$ is birational.

- **Kollár’s method:** taking a sub-pencil $\Lambda \subset |kK_V|$, one gets a surjective morphism $f : V \rightarrow \Gamma \cong \mathbb{P}^1$. One has the inclusion $\mathcal{O}(1) \hookrightarrow f_*\omega^k_V$ and then, for any $p \geq 5$,

$$f_*\omega^p_{V/\Gamma} \otimes \mathcal{O}(1) \hookrightarrow f_*\omega^{(2p+1)k+p}_V.$$

Since the 5-canonical map of the general fiber is birational and by the semi-positivity of $f_*\omega^p_{V/\Gamma}$, one sees that $\varphi_{11k+5}$ is birational by simply taking $p = 5$. 
• Proved by M. Chen in 2004:

**Theorem**

Let $V$ be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then $\varphi_m$ is birational for all $m \geq 5k + 6$. 
II–1.3. Improved form of Kollár’s Theorem

- Proved by M. Chen in 2004:

**Theorem**

Let $V$ be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then $\varphi_m$ is birational for all $m \geq 5k + 6$.

- Kollár’s method + the geometry of linear systems
II–1.4. The case $r_X \geq 2$

- Suppose $\chi(O_X) < 0$. Reid’s Riemann-Roch formula implies $P_2(X) \geq 4$. Hence the question is solvable by Kollár’s theorem.
II-1.4. The case \( r_X \geq 2 \)

- Suppose \( \chi(\mathcal{O}_X) < 0 \). Reid’s Riemann-Roch formula implies \( P_2(X) \geq 4 \). Hence the question is solvable by Kollár’s theorem.
- Suppose that \( P_m \geq 2 \) for some \( m \leq 12 \), one applies Kollár’s method as well.
II-1.4. The case $r_X \geq 2$

- Suppose $\chi(O_X) < 0$. Reid’s Riemann-Roch formula implies $P_2(X) \geq 4$. Hence the question is solvable by Kollár’s theorem.
- Suppose that $P_m \geq 2$ for some $m \leq 12$, one applies Kollár’s method as well.
- The remain situation: $\chi(O_X) \geq 0$ and $P_k(X) \leq 1$ for all $2 \leq k \leq 12$. Key Inequality reads:

$$2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi(O_X) + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13},$$

which directly implies that $\chi(O_X) \leq 8$, $P_{13} \leq 7$. 
II-1.5. Boundedness Results

Now $B^{12}(X)$ has finite possibilities and $B^{12}(X) \subseteq B(X)$. So $B(X)$ has finite possibilities.
II–1.5. Boundedness Results

- Now $B^{12}(X)$ has finite possibilities and $B^{12}(X) \succeq B(X)$. So $B(X)$ has finite possibilities.
- Chen-Chen 2010-2015:

**Theorem**

Let $X$ be a minimal projective 3-fold of general type. Then

1. $K^3_X \geq \frac{1}{1680};$
2. $\varphi_{m,X}$ is birational for all $m \geq 61;$
3. $P_{12} \geq 1$ and $P_{24} \geq 2.$
4. $K^3_X \geq \frac{1}{420}$ (optimal) if $\chi(O_X) \leq 1.$
II–1.6. Explicit Classifications

- Define the **pluricanonical section index** $\delta(X)$ to be the minimal integer so that $P_\delta \geq 2$.

**Theorem**

Let $X$ be a minimal projective 3-fold of general type. Then

1. $\delta(X) \leq 18$;
2. $\delta(X) = 18$ if and only if $\mathbb{B}(X) = \{B_{2a}, 0, 2\}$;
3. $\delta(X) \neq 16, 17$;
4. $\delta(X) = 15$ if and only if $\mathbb{B}(X)$ belongs to one of the types in [CC3, Table F–1];
5. $\delta(X) = 14$ if and only if $\mathbb{B}(X)$ belongs to one of the types in [CC3, Table F–2];
6. $\delta(X) = 13$ if and only if $\mathbb{B}(X) = \{B_{41}, 0, 2\}$.
II–1.7. The Up-to-date Result!

- Recently M. Chen showed $r_3 \leq 57$ on the basis of above classifications. Therefore, $27 \leq r_3 \leq 57$. 

"Jungkai A. Chen & Meng Chen (NTU & Fudan)"

"On Explicit Aspect of Pluricanonical Maps of Projective Varieties"
II–1.7. The Up-to-date Result!

• Recently M. Chen showed $r_3 \leq 57$ on the basis of above classifications. Therefore, $27 \leq r_3 \leq 57$.
• For 3-folds with $\delta = 1$, M. Chen proved the following optimal results:

**Theorem**

*Let $X$ be a minimal projective 3-fold of general type with $p_g(X) \geq 2$. Then*

1. $K_X^3 \geq \frac{1}{3}$;
2. $\varphi_{8,X}$ is birational onto its image.
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1. $K_X^3 \geq \frac{1}{3}$;
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For 3-folds with $\delta = 2$, Chen-Chen proved that $r_s(X) \leq 11$ (optimal).
II–2.1. \( \mathbb{Q} \)-Fano 3-folds

A normal projective 3-fold \( X \) is called a \underline{weak \( \mathbb{Q} \)-Fano 3-fold} (resp. \underline{\( \mathbb{Q} \)-Fano 3-fold}) if the anti-canonical divisor \( -K_X \) is nef and big (resp. ample). A \underline{canonical} (resp. \underline{terminal}) weak \( \mathbb{Q} \)-Fano 3-fold is a weak \( \mathbb{Q} \)-Fano 3-fold with at worst canonical (resp. terminal) singularities.
II–2.1. **Q-Fano 3-folds**

- A normal projective 3-fold $X$ is called a **weak $Q$-Fano 3-fold** (resp. **$Q$-Fano 3-fold**) if the anti-canonical divisor $-K_X$ is nef and big (resp. ample). A **canonical** (resp. **terminal**) weak $Q$-Fano 3-fold is a weak $Q$-Fano 3-fold with at worst canonical (resp. terminal) singularities.

- Take the weighted basket

$$\mathcal{B}(X) = \{ B_X, P_{-1}, \chi(\mathcal{O}_X) \}.$$ 

By the duality and the vanishing of higher cohomology, we always have $\chi_m = -P_{-(m-1)}$ for all $m \geq 2$. Hence the basket theory has a parallel version in Fano case.
In 2008, Chen-Chen applied the basket theory to prove the following theorem:

**Theorem**

Let $X$ be a terminal (or canonical) weak $\mathbb{Q}$-Fano 3-fold. Then

1. $P_{-4} > 0$ with possibly one exception of a basket of singularities;
2. $P_{-6} > 0$ and $P_{-8} > 1$;
3. $-K^3_X \geq \frac{1}{330}$. Furthermore $-K^3_X = -\frac{1}{330}$ if and only if the basket of singularities is $\{(1, 2), (2, 5), (1, 3), (2, 11)\}$.

The above theorem is optimal according to Fletcher: $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$. 
• **M. Chen** started to study the constant $m_3$ in 2011.
II–2.3. The Anti-pluricanonical Birationality

- M. Chen started to study the constant $m_3$ in 2011.
- Chen-Jiang proved in 2016:

**Theorem**

Let $X$ be a terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then $\varphi_{-m,X}$ is birational for all $m \geq 39$.

**Theorem**

Let $X$ be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $\varphi_{-m,X}$ is birational for all $m \geq 97$. 
II–2.4. The Anti-pluricanonical Birationality

• “$m_3 \leq 97$” is far from being optimal!
II–2.4. The Anti-pluricanonical Birationality

- “$m_3 \leq 97$” is far from being optimal!
- Chen–Jiang proved in 2017:

**Theorem**

Let $V$ be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then, for any K-Mori fiber space $Y$ of $V$, $\varphi_{-m,Y}$ is birational for all $m \geq 52$. 
II–3. The Noether inequality for algebraic 3-folds

II–3.1. Recall–The Surface Geography

- **General strategy of the geography**

![Diagram](image)

- Miyaoka-Yau inequality
- The Noether inequality

\[ c_1^2 \]

\[ O \]
II–3.2. The Noether Inequality

• There is no effective 3-dimensional analogy of Miyaoka-Yau inequality \( K_S^2 \leq 9 \chi(\mathcal{O}_S) \), since \( -\infty < \chi(\mathcal{O}_X) < +\infty \).
II–3. The Noether inequality for algebraic 3-folds

II–3.2. The Noether Inequality

- There is **no effective** 3-dimensional analogy of Miyaoka-Yau inequality “$K_S^2 \leq 9\chi(\mathcal{O}_S)$”, since $-\infty < \chi(\mathcal{O}_X) < +\infty$.

- **Seek for the Noether inequality!**
• $X$ minimal, the Cartier index $r_X \geq 1$.

$$X \text{ is Gorenstein } \iff r_X = 1$$

$\{\text{smooth minimals}\} \subset \{\text{Gorenstein minimals}\} \subset \{\text{General minimals}\}$
II–3.3. History of 3-Dimensional Noether Inequality

• $X$ minimal, the Cartier index $r_X \geq 1$.

$X$ is Gorenstein $\iff r_X = 1$

$\{\text{smooth minimals}\} \subset \{\text{Gorenstein minimals}\} \subset \{\text{General minimals}\}$

• The possible Noether type inequality is of the form:

$$K_X^3 \geq ap_g(X) - b$$

$a, b \in \mathbb{Q}_{>0}$. 
II–3.4. The Noether Inequality for Gorenstein Minimal 3-folds

- **Kobayashi (1992)**: an infinite series of examples of canonically polarized 3-folds satisfying $K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3}$. 
II–3.4. The Noether Inequality for Gorenstein Minimal 3-folds

- **Kobayashi (1992)**: an infinite series of examples of canonically polarized 3-folds satisfying $K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3}$.
- **M. Chen (2004)**: $K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3}$ for canonically polarized 3-folds.
II–3.4. The Noether Inequality for Gorenstein Minimal 3-folds

- **Kobayashi (1992)**: an infinite series of examples of canonically polarized 3-folds satisfying \( K_X^3 = \frac{4}{3} p_g(X) - \frac{10}{3} \).
- **M. Chen (2004)**: \( K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3} \) for canonically polarized 3-folds.
- **Catanese-Chen-Zhang (2006)**: \( K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3} \) for smooth minimal 3-folds of general type.
II–3.4. The Noether Inequality for Gorenstein Minimal 3-folds

- **Kobayashi (1992)**: an infinite series of examples of canonically polarized 3-folds satisfying $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$.

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- **Catanese-Chen-Zhang (2006)**: $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for smooth minimal 3-folds of general type.

- **Chen-Chen (2015)**: $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ for Gorenstein minimal 3-folds of general type.
• May always assume $p_g(X) \geq 2$, since $K_X^3 > 0$. So $\varphi|_{K_X}$ is non-trivial.
• May **always assume** $p_g(X) \geq 2$, since $K_X^3 > 0$. So $\varphi_{|K_X|}$ is non-trivial.

• Set up for $\varphi_1 = \varphi_{|K_X|}$. Set $d_X = \dim(\Gamma)$.
II–3.6. The main statement

**Theorem**

Let $X$ be a minimal projective 3-fold of general type. Assume that one of the following holds:

- $d_X \geq 2$; or
- $d_X = 1$ and $|K_X|$ is not composed with a rational pencil of $(1, 2)$-surfaces; or
- $d_X = 1$, $|K_X|$ is composed with a rational pencil of $(1, 2)$-surfaces, and either $p_g(X) \leq 4$ or $p_g(X) \geq \frac{2}{\text{glct}(1,2)} + 1$.

Then the inequality

$$K_X^3 \geq \frac{4}{3} p_g(X) - \frac{10}{3}$$

holds.
II–3.7. The Noether Inequality for Algebraic 3-folds

- János Kollár: $\text{glct}(1, 2) \geq \frac{1}{10}$ (optimal).
II–3.7. The Noether Inequality for Algebraic 3-folds

- János Kollár: $\text{glct}(1, 2) \geq \frac{1}{10}$ (optimal).
- Chen-Chen-Jiang (2018) proved the following:

**Theorem**

Let $X$ be a minimal projective 3-fold of general type and either $p_g(X) \leq 4$ or $p_g(X) \geq 21$. Then the inequality holds:

$$K^3_X \geq \frac{4}{3} p_g(X) - \frac{10}{3}.$$

**Corollary**

The inequality $K^3_X \geq \frac{4}{3} p_g(X) - \frac{10}{3}$ holds except for finite number of families of 3-folds of general type.
II–3.8. Conjecture A

• The 3D-Noether Inequality

Conjecture

The inequality $K^3 \geq \frac{4}{3} p_g - \frac{10}{3}$ holds for all minimal 3-folds of general type with $5 \leq p_g \leq 20$. 
II-3. The Noether inequality for algebraic 3-folds

II–3.8. Conjecture A

- The 3D-Noether Inequality

**Conjecture**

The inequality $K^3 \geq \frac{4}{3} p_g - \frac{10}{3}$ holds for all minimal 3-folds of general type with $5 \leq p_g \leq 20$.

- Projective varieties with very large canonical volumes. For $n \geq 2$, recall:

  $$r_n = \max \{ r_s(X) \mid X \text{ is a } n\text{-fold of general type} \};$$

  $$r_n^+ = \max \{ r_s(X) \mid X \text{ is a } n\text{-fold of general type with } p_g > 0 \};$$
II–3. The Noether inequality for algebraic 3-folds

II–3.8. Conjecture A

- The 3D-Noether Inequality

**Conjecture**

The inequality $K^3 \geq \frac{4}{3} p_g - \frac{10}{3}$ holds for all minimal 3-folds of general type with $5 \leq p_g \leq 20$.

- Projective varieties with very large canonical volumes. For $n \geq 2$, recall:

  \[ r_n = \max \{ r_s(X) \mid X \text{ is a n-fold of general type} \}; \]

  \[ r_n^+ = \max \{ r_s(X) \mid X \text{ is a n-fold of general type with } p_g > 0 \}; \]

- By definition, one has $r_n^+ \leq r_n$. 
II–3.9. Conjecture B and Conjecture C

- Conjecture B.

Conjecture

There exists a function $K(n)$ such that $r_s(X) \leq r_{n-1}$ holds for any $n \geq 5$ and for any minimal projective $n$-fold $X$ with $K_X^n > K(n)$. 
II–3.9. Conjecture B and Conjecture C

- Conjecture B.

**Conjecture**

There exists a function $K(n)$ such that $r_s(X) \leq r_{n-1}^+ \leq 1$ holds for any $n \geq 5$ and for any minimal projective $n$-fold $X$ with $K^n_X > K(n)$.

- Conjecture C.

**Conjecture**

There exists a function $L(n)$ such that $r_s(X) \leq r_{n-1}^+ \leq 1$ holds for any $n \geq 6$ and for any minimal projective $n$-fold $X$ of general type with $p_g > L(n)$.
II-3. The Noether inequality for algebraic 3-folds

Thank you very much!