Pairs of Invariants of Normal Surface Singularities

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ICM, Rio de Janeiro, August 4, 2018

$(X,o) \quad = \quad$ analytic germ of a normal surface singularity / $\mathbb{C}$

$M := X \cap S^{2N-1}_\varepsilon$ = the link of (X,o) (where $(X,o)\subset(\mathbb{C}^N,o)$), oriented 3–manifold

Fix a good resolution $\widetilde{X} \xrightarrow{\pi} X$,

- exceptional curve: $\pi^{-1}(o) = E = \cup_{v\in\mathcal{V}} E_v$, E_v irreducible
- dual graph: Γ
- intersection form $(E_v, E_u)_{u,v}$
- $L := (\mathbb{Z}\langle E_v\rangle_v, (\,,\,)) = H_2(\widetilde{X},\mathbb{Z})$ negative definite lattice
- $L' = H^2(\widetilde{X},\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z}) \subset L\otimes\mathbb{Q}$ dual lattice
- discriminant group $H := L'/L$, $\quad \widehat{H}$ = Pontryagin dual of H
- $\theta : H \to \widehat{H}$ natural isomorphism $[l'] \mapsto \theta([l']) := e^{2\pi i(l',\cdot)}$
- The Lipman cone: $\mathcal{S}' := \{l' \in L' : (l', E_v) \le 0 \ \forall v\}$

Assume:

M is a **rational homology sphere** ($H_1(M, \mathbb{Z})$ is finite)

$\Leftrightarrow L'/L = H = H_1(M, \mathbb{Z})$,

$\Leftrightarrow \Gamma$ is a tree and $E_v \simeq \mathbb{P}^1$ for all v

Analytic invariants $\Leftrightarrow$ local algebra $\mathscr{O}_{X,o}$, or
$\Leftrightarrow$ complex analytic manifold $\widetilde{X}$

Topological invariants $\Leftrightarrow$ topological/smooth invariants of M, or
$\Leftrightarrow$ combinatorial invariants of Γ, or of L

Goal: connect the analytic and topological invariants

Question: how the analytic and topological types
influence each other ?

Some invariants of the analytic type:

- Cohomology of holomorphic line bundles on $\widetilde{X}$
- Cohomology of *natural* line bundles on $\widetilde{X}$ [Okuma], [N.]
- H–equivariant multivariable divisorial multifiltration $\mathscr{F}$ and family of linear subspace arrangements, both indexed by L' [Campillo, Delgado, Gusein–Zade]
- H–equivariant multivariable Hilbert and Poincaré series of $\mathscr{F}$
- Principle cycles (cut out by sections of line bundles or functions)
- geometric genus $p_g = h^1(\mathscr{O}_{\widetilde{X}})$, H–equivariant geometric genus

Problem: determine these invariants,
find computational methods (e.g. surgery formulae)
Question: when are they topological ? (computable from Γ or M)

Some invariants of the topological type:

- Seiberg–Witten invariant of the link
- Heegaard–Floer (co)homology of the link [Ozsváth–Szabó], or Monopole Floer (co)homology of the link [Kronheimer–Mrowka]
- Lattice (co)homology, graded roots [N.]
- a topological family of linear subspace arrangements indexed by L' [N.]
- H–equivariant multivariable zeta–function (series)

Problem: determine these invariants,
find computational methods (e.g. surgery formulae)
Question: what are their peculiar/additional properties
(valid for singularity links)?

Some definitions of the analytic part:

'Natural' line bundles (with given Chern class):

the cohomological exponential exact sequence splits uniquely extending the natural $L \ni l \mapsto \mathcal{O}(l)$ split over integral cycles L

$$
\begin{array}{ccccccccc}
 & & & & & & L & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) & \longrightarrow & \mathrm{Pic}(\widetilde{X}) & \underset{\mathcal{O}}{\overset{c_1}{\rightleftarrows}} & L' & \longrightarrow & 0.
\end{array}
$$

This provides to each rational cycle $l' \in L' = H^2(\widetilde{X}, \mathbb{Z})$ a line bundle $\mathcal{O}(l') \in \mathrm{Pic}(\widetilde{X})$, whose first Chern class c_1 is l'.

Example of natural line bundle:

Let $c:(X^a,o)\to(X,o)$ be the **universal abelian cover** of (X,o),
$\pi^a:\widetilde{X^a}\to X^a$ the normalized pullback of the resolution π by c,

$$\begin{array}{ccc} \widetilde{X^a} & \xrightarrow{\pi^a} & (X^a,o) \\ \Big\downarrow \widetilde{c} & & \Big\downarrow c \\ \widetilde{X} & \xrightarrow{\pi} & (X,o) \end{array}$$

The action of H on (X^a,o) lifts to $\widetilde{X^a}$ and gives an H-eigensheaf decomposition into natural line bundles

$$\widetilde{c}_*\mathcal{O}_{\widetilde{X^a}}=\bigoplus_{l'\in\mathfrak{C}}\mathcal{O}(-l'),\quad \text{with } \mathfrak{C}=\{\sum l'_vE_v\in L',\ 0\leq l'_v<1\}$$

$$\#\,\mathfrak{C}=\#\,H$$

Notations: $\forall h\in H$ set $r_h\in\mathfrak{C}$ with $[r_h]=h\in L'/L=H$

$$\widetilde{c}_* \mathcal{O}_{\widetilde{X^a}} = \bigoplus_{h\in H} \mathcal{O}(-r_h)$$

$$p_g(X^a, o) = \sum_h h^1(\mathcal{O}(-r_h))$$

equivariant geometric genus: $\{h^1(\mathcal{O}(-r_h))\}_h$

H–invariant, geometric genus:

$$r_h = 0,\ \mathcal{O}(0) = \mathcal{O}_{\widetilde{X}},\ h^1(\mathcal{O}_{\widetilde{X}}) = p_g(X, o)$$

For fixed π, $\mathscr{O}_{X^a,o}$ inherits the L'–**indexed divisorial filtration**

$$\mathscr{F}(l') := \{f \in \mathscr{O}_{X^a,o} \,|\, \mathrm{div}(f \circ \pi^a) \geq \widetilde{c}^*(l')\}.$$

$\mathfrak{h}(l') =$ dimension of the $\theta([l'])$-eigenspace of $\mathscr{O}_{X^a,o}/\mathscr{F}(l')$.

The **equivariant divisorial Hilbert series** is

$$\mathscr{H}(\mathbf{t}) = \sum_{l'=\sum l_v E_v \in L'} \mathfrak{h}(l') t_1^{l_1} \cdots t_s^{l_s} = \sum_{l' \in L'} \mathfrak{h}(l') \mathbf{t}^{l'} \in \mathbb{Z}[[L']].$$

$h \mapsto \mathscr{H}_h(\mathbf{t}) = \sum_{[l']=h} \mathfrak{h}(l') \mathbf{t}^{l'} \leftrightarrow H$-eigensheaf decomposition.

$\mathscr{H}(\mathbf{t}) = \sum_{h \in H} \mathscr{H}_h(\mathbf{t})$

$\mathscr{H}_0(\mathbf{t}) = \sum_{l \in L} \mathfrak{h}(l) \mathbf{t}^l \quad \leftrightarrow H$-invariants,

$=$ the *Hilbert series* of $\mathscr{O}_{X,o}$ associated with the L–indexed $\{E_v\}_v$–divisorial filtration.

Equivariant Poincaré series: ('multi–graded version')

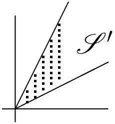
$$\mathscr{P}(\mathbf{t}) = -\mathscr{H}(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}) = \sum_{l' \in \mathscr{S}'} \mathfrak{p}(l')\mathbf{t}^{l'} \in \mathbb{Z}[[\mathscr{S}']]$$

$$\mathscr{P}(\mathbf{t}) = \sum_{h \in H} \mathscr{P}_h(\mathbf{t})$$

Fact:

$$\mathscr{P}(\mathbf{t}) \ \Leftrightarrow \ \mathscr{H}(\mathbf{t})$$

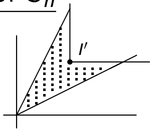
advantage of $\mathscr{P}$: it is supported on the positive cone $\mathscr{S}'$



Aside from generalized Ehrhart theory:

Fix a series $S(\mathbf{t})=\sum_{l'\in L'} s(l')\mathbf{t}^{l'}$, supported on a positive cone,
Decompose as $S=\sum_{h\in H} S_h$, where $S_h(\mathbf{t})=\sum_{[l']=h} s(l')\mathbf{t}^{l'}$,
Consider the <u>counting function of the coefficients of S_h</u>

$$Q_h(l')=\sum_{a\in L,\, a\ngeq 0} s(l'+a) \quad ([l']=h).$$



Assume that there exist: a real cone $\mathscr{K}\subset L'\otimes\mathbb{R}$, $l'_*\in\mathscr{K}$,
a sublattice $\widetilde{L}\subset L$ of finite index, and
a quasipolynomial $\mathfrak{Q}_h(l)$ $(l\in\widetilde{L})$ such that

$$Q_h(l+r_h)=\mathfrak{Q}_h(l) \quad \forall \;\; l+r_h\in (l'_*+\mathscr{K})\cap(\widetilde{L}+r_h).$$

Then we say that $S_h(\mathbf{t})$ admits a **quasipolynomial in $\mathscr{K}$**, namely $\mathfrak{Q}_h(l)$, and also an (equivariant, multivariable) **periodic constant** associated with $\mathscr{K}$, which is defined as

$$\mathrm{pc}^{\mathscr{K}}(S_h(\mathbf{t})):=\mathfrak{Q}_h(0).$$

The quasipolynomial of $\mathscr{P}$:

Theorem: The counting function of $\mathscr{P}$ satisfies for $l' = l + r_h \in L'$

$$\sum_{a\in L,\, a\not\geq 0} \mathfrak{p}(l'+a) = \mathfrak{h}(l')$$

$$= -h^1(\widetilde{X}, \mathscr{O}(-l')) + \chi(l) - (r_h, l) + h^1(\widetilde{X}, \mathscr{O}(-r_h)).$$

Here $\chi(l) = -(l, l+K)/2$, $K = K_{\widetilde{X}} \in L'$ is the canonical class.

If $l' \in -K + \mathscr{S}'$ then by the vanishing $h^1(\mathscr{O}(-l')) = 0$

$$\sum_{a\in L,\, a\not\geq 0} \mathfrak{p}(l'+a) = \chi(l) - (r_h, l) + h^1(\widetilde{X}, \mathscr{O}(-r_h)).$$

This quadratic function is the **quasipolynomial of $\mathscr{P}$ in $\mathscr{K} = \mathscr{S}'_{\mathbb{R}}$**, with **periodic constant**

$$\mathrm{pc}^{\mathscr{S}'}(\mathscr{P}_h(\mathbf{t})) = h^1(\widetilde{X}, \mathscr{O}(-r_h)).$$

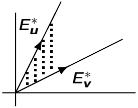
The topological analogue:

Set $E^*_v \in L'$ defined by $(E^*_v, E_u) = -\delta_{v,u}$ for all $u \in \mathscr{V}$.

Remark: $L' = \mathbb{Z}\langle E^*_v\rangle_v$, and $\mathscr{S}' = \mathbb{Z}_{\geq 0}\langle E^*_v\rangle_v$.

The topological multivariable ('zeta') series, supported on $\mathscr{S}'$, is the Taylor expansion $Z(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'}$ at the origin of

$$\prod_{v\in\mathscr{V}} (1-\mathbf{t}^{E^*_v})^{\delta_v - 2} \qquad (\delta_v = \text{valency of } v \text{ in } \Gamma)$$



Seiberg–Witten invariants of the link M.

Let $\widetilde{\sigma}_{can}$ be the *canonical spin^c–structure on* $\widetilde{X}$ identified by $c_1(\widetilde{\sigma}_{can}) = -K$, and let $\sigma_{can} \in \mathrm{Spin}^c(M)$ be its restriction to M.

$\mathrm{Spin}^c(M)$ is an H–torsor with action denoted by $*$.

We denote by $\mathfrak{sw}_\sigma(M) \in \mathbb{Q}$ the *Seiberg–Witten invariants* of M indexed by the *spin^c*–structures $\sigma \in \mathrm{Spin}^c(M)$.

The quasipolynomial of Z:

Theorem [N.] The counting function of $Z_h(\mathbf{t})$ in the cone $S'_{\mathbb{R}}$ admits the (quasi)polynomial

$$\mathfrak{Q}_h(l) = -\frac{(K+2r_h+2l)^2+|\mathcal{V}|}{8} - \mathfrak{sw}_{-h*\sigma_{can}}(M),$$

whose periodic constant is

$$(*)\quad \mathrm{pc}^{S'_{\mathbb{R}}}(Z_h(\mathbf{t})) = \mathfrak{Q}_h(0) = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K+2r_h)^2+|\mathcal{V}|}{8}.$$

The right hand side of $(*)$ with opposite sign is called the r_h*–normalized Seiberg–Witten invariant* of M.

The analogy between $\mathscr{P}(\mathbf{t})$ and $Z(\mathbf{t})$ culminates in the fact that for many analytic types $\mathscr{P} = Z$, hence $\mathscr{P}$ and $\mathscr{H}$ have topological descriptions.

Theorem: $\mathscr{P}(\mathbf{t}) = Z(\mathbf{t})$ holds in the following cases:
(a) *rational singularities* [Campillo—Delgado—Gusein-Zade]
(b) *minimally elliptic singularities* [N.]
(c) *splice quotient singularities* [N.] (this includes (a), (b) and the *weighted homogeneous* case as well).
(Splice quotient germs were introduced by Neumann–Wahl, later redefined by Okuma.)

Remark: $\mathscr{P} = Z$ is not always true (e.g. for superisolated germs).
Question: What is the limit of the identity $\mathscr{P}(\mathbf{t}) = Z(\mathbf{t})$?
Answer: $\mathscr{P}(\mathbf{t}) = Z(\mathbf{t})$ iff (X, o) is *splice quotient*. [N.]

Recall: $\mathrm{pc}^{\mathscr{S}'}(\mathscr{P}_h)$=$h$–part of the equivariant geometric genus,
$\mathrm{pc}^{\mathscr{S}'}(Z_h)$=$r_h$-normalized Seiberg–Witten invariant.

The identity of the right hand sides was conjectured/studied even before the appearance of $\mathscr{P}=Z$ identity: this is the **Seiberg–Witten Invariant Conjecture** of Nicolaescu-N., as extension of the *Casson Invariant Conjecture* of Neumann-Wahl

We say that (X,o) satisfy the *equivariant* SWIC if for any $h\in H$

$$h^1(\widetilde{X},\mathscr{O}(-r_h))=-\mathfrak{sw}_{-h*\sigma_{can}}(M)-\frac{(K+2r_h)^2+|\mathscr{V}|}{8}$$

This (automatically) extends to arbitrary natural line bundles as

$$h^1(\widetilde{X},\mathscr{O}(l'))=-\mathfrak{sw}_{[l']*\sigma_{can}}(M)-\frac{(K-2l')^2+|\mathscr{V}|}{8}$$

We say that (X,o) satisfies the SWIC (for $h=0$) if

$$p_g(X,o)=-\mathfrak{sw}_{\sigma_{can}}(M)-\frac{K^2+|\mathscr{V}|}{8}.$$

Remark: The (equivariant) SWIC might hold even if $\mathscr{P} = Z$ fails.

Theorem: The equivariant SWIC holds in the following cases:

(a) *rational singularities* [Nicolaescu–N.]
(b) *weighted homogeneous singularities* [Nicolaescu–N.]
(c) *splice quotient singularities* [N.]

Additionally, the SWIC (for $h = 0$) folds for
(a) *suspensions* $\{f(x,y) + z^N = 0\}$ with f irred. [Nicolaescu–N.]
(b) *hypersurface Newton non–degenerate germs* [N.–Sigurdsson]
(c) *superisolated singularities with one cusp*
[Bobadilla–Luengo–Melle-Hernández–N.] and [Borodzik–Livingston]

Since the identity of the SWIC is stable with respect to equisingular deformations, the SWIC remains valid for such deformations of any of the above cases. (The same is true for the identity $\mathscr{P} = Z$.)

SWIC might fail, e.g. for some **superisolated singularities** but still it indicates several information...

Classification problem of projective plane curves
$C = \{f(x,y,z)=0\} \subset \mathbb{P}^2$, $\deg(f)=d$
$\mathrm{Sing}(C)=\{p_1,...,p_r\}$ has local knots $K_1,...,K_r$, $K_i \subset S^3$.

Question: For fixed d what local singularity types can be realized?

Construction of superisolated surface singularities
$\widetilde{f} := f+$generic $>d$ terms, $(X,o)=\{\widetilde{f}=0\}\subset(\mathbb{C}^3,0)$ SI sing.
$p_g = d(d-1)(d-2)/6$, $\quad M = S^3_{-d}(K_1\#\cdots\#K_r)$
M $\mathbb{Q}HS^3 \Leftrightarrow C$ is rational cuspidal
SWIC $\Leftrightarrow$ interesting distribution property of the semigroups of local singularity types $K_i \subset S^3 \Rightarrow$ obstructions for classification of curves

Reinterpretations, generalizations created a lot of activity....
[Bobadilla–Luengo–Melle-Hernández–N.], [Borodzik–Livingston], [Bodnár–Colaria–Golla], [Borodzik–Hom], [Borodzik–Moe], ...

More analogy between $\mathscr{P}(\mathbf{t})$ and $Z(\mathbf{t})$: Surgery formulae
(independently of $Z = \mathscr{P}$, for any topological/analytic type)

Topological surgery formula (any Γ, any $I \subset \mathscr{V}$ fixed):
$Inv(\Gamma)$=normalized Seiberg–Witten invariant associated with $M(\Gamma)$,

$$Inv(\Gamma) - Inv(\Gamma \setminus \{I\}) = -\mathrm{pc}(Z|_{t_v=1 \text{ for all } v \notin I})$$

[Braun-N.], [László-Nagy-N.] (this is <u>not</u> the surgery given by 'exact triangles of topological cohomology theories', e.g. via $HF^{\pm}$ theory)
Analytic surgery formula ('almost any' (X, o), any $I \subset \mathscr{V}$ fixed):
$Inv(\widetilde{X}(\Gamma))$=equivariant geometric genus associated with $\widetilde{X}$,

$$Inv(\widetilde{X}(\Gamma)) - Inv(\widetilde{X}(\Gamma \setminus \{I\})) = \mathrm{pc}(\mathscr{P}|_{t_v=1 \text{ for all } v \notin I})$$

[Okuma], [Nagy-N.]
(analytic theory is 'rigid', surgery formulae are surprising, rare....)

Linear subspace arrangements indexed by L', as 'lifts' of the series $\mathscr{P}(\mathbf{t})$ and $Z(\mathbf{t})$

Fix a Chern class $l' \in L'$. For any analytic type supported on Γ

$$\forall v: \ H^0(\mathscr{O}_{\widetilde{X}}(-l'-E_v)) \subset H^0(\mathscr{O}_{\widetilde{X}}(-l')) \quad \infty-\text{dim'l arrangement}$$

truncation:

$$A(l') := (\mathscr{F}(l')/\mathscr{F}(l'+E))_{\theta([l'])} = H^0(\mathscr{O}_{\widetilde{X}}(-l'))/H^0(\mathscr{O}_{\widetilde{X}}(-l'-E))$$

and for any $v \in \mathscr{V}$ a linear subspace in $A(l')$

$$A_v(l') := H^0(\mathscr{O}_{\widetilde{X}}(-l'-E_v)/H^0(\mathscr{O}_{\widetilde{X}}(-l'-E))$$

$\mathscr{A}_{\rm an}(l') := \{A_v(l')\}_v$ in $A(l')$ = '*analytic*' subspace arrangement.
[Campillo—Delgado—Gusein-Zade]

Fact: $\mathscr{A}_{\rm an}(l')$ embeds into another arrangement
Set $T(l') := H^0(\mathscr{O}_E(-l'))$, and for any $v \in \mathscr{V}$ set

$$T_v(l') := H^0(\mathscr{O}_{E-E_v}(-l'-E_v)) \subset T(l').$$

Fact. The linear subspace arrangement $\mathscr{A}_{\rm top}(l') := \{T_v(l')\}_v$ in $T(l')$ is *topological*, it depends only on Γ.

$$\begin{array}{ccc} A(l') & \hookrightarrow & T(l') \\ \cup & & \cup \\ A_v(l') = A(l') \cap T_v(l') & \hookrightarrow & T_V(l') \end{array} \qquad (*)$$

Fixed topological type Γ $\Rightarrow$ $\mathscr{A}_{\rm top}(l')$
Any analytic type supported by $\Gamma \Rightarrow$ $A(l') \subset T(l')$ (with 'moving dimension and position') which determines $\mathscr{A}_{\rm an}(l')$ by $(*)$

Theorem:

$$\mathscr{P}(\mathbf{t}) = \sum_{l'} \chi_{\text{top}}(\,\mathbb{P}(A(l')\setminus \cup_v A_v(l'))\,)\cdot \mathbf{t}^{l'}$$

$$Z(\mathbf{t}) = \sum_{l'} \chi_{\text{top}}(\,\mathbb{P}(\,T(l')\setminus \cup_v T_v(l'))\,)\cdot \mathbf{t}^{l'}$$

E.g.

- (X,o) rational $\Rightarrow \mathscr{A}_{an}(l') = \mathscr{A}_{top}(l')\ \forall\ l' \Rightarrow \mathscr{P} = Z$

- Examples of splice quotient: $A(l') \pitchfork \mathscr{A}_{top}(l')\ \forall\ l' \Rightarrow \mathscr{P} = Z$

(Recall: $\mathscr{P} = Z \Rightarrow$ SW–characterization of all $h^1(\mathscr{O}(l')))$

- $P \neq Z \Rightarrow A(l')$ 'small' and it has 'bad' position in $\mathscr{A}_{top}(l')$

Task: Characterize all linear subspace arrangements, indexed by some lattice, which might appear as $\{\mathscr{A}_{top}(l')\}_{l'}$.

Task: For any fixed $\{\mathscr{A}_{top}(l')\}_{l'}$, characterize all linear subspace arrangements (up to isotopy, or even the corresponding moduli space in some flag manifolds) which might appear as $\{\mathscr{A}_{an}(l')\}_{l'}$ in $\{\mathscr{A}_{top}(l')\}_{l'}$.

Task: Lift all the invariants above to the 'motivic' level in the Grothendieck ring. E.g., at topological level, set

$$Z(\mathbb{L},\mathbf{t}) = \sum_{l'\in\mathscr{S}'} [\mathbb{P}(T(l')\setminus\cup_v T_v(l'))]\cdot \mathbf{t}^{l'}.$$

Then (with $\mathscr{V}$=vertices, $\mathscr{E}$=edges of Γ)

$$Z(\mathbb{L},\mathbf{t}) = \frac{\prod_{(u,v)\in\mathscr{E}} (1-\mathbf{t}^{E_u^*}-\mathbf{t}^{E_v^*}+\mathbb{L}\mathbf{t}^{E_u^*+E_v^*})}{\prod_{v\in\mathscr{V}} (1-\mathbf{t}^{E_v^*})(1-\mathbb{L}\mathbf{t}^{E_v^*})}.$$

There is a cohomology theory behind the topological part [N.]:

the lattice cohomology $\mathbb{H}^*(M,\sigma)$, $\sigma\in \mathrm{Spin}^c(M)$

- $L\otimes\mathbb{R}$ has a cellular decomposition into cubes.

  0–cubes $= L$, 1–cubes $=$ 'segments' $[l, l+E_v]$, etc.
- For a characteristic element $k\in L'$, set $\chi_k(l)=-(l,l+k)/2$,

(Note $\mathrm{Char}=K+2L'$ and $\mathrm{Char}/2L=\mathrm{Spin}^c(M)$)

- S_n = the union of all q–cubes in $\{l\in L\otimes\mathbb{R} : \chi_k(l)\leq n\}$.

Definition [N.]

$$\mathbb{H}^p(\Gamma,k):=\oplus_n H^p(S_n,\mathbb{Z}).$$

Each $\mathbb{H}^q$ ($q\geq 0$) is a graded $\mathbb{Z}[U]$–module.
U–action = cohomological restrictions given by the inclusions

$$\ldots\subset S_{n-1}\subset S_n\subset S_{n+1}\subset\ldots$$

Properties of $\mathbb{H}^*$ [N.]:

- it is an invariant of M
- it characterizes families of singularities (rational, elliptic,...)
- $\exists$ cohomological long exact sequences (associated with surgeries)
- vanishing : $\#$ of 'bad' vertices $\leq n \Rightarrow \ \mathbb{H}^q = 0$ for $q \geq n$
- reduction (to the – much smaller – sublattice of 'bad' vertices)
- conjecture: $\mathbb{H}^* \Rightarrow HF$ (true for $\#$ of 'bad' vertices ≤ 2)
- 'reduced' $\mathbb{H}^* = 0 \Leftrightarrow M$ is an L–space (reduced HF =0)
- normalised Euler characteristic of $\mathbb{H}^*$ is the Seiberg–Witten inv
- multi-graded Euler characteristic $\Rightarrow Z(\mathbf{t})$
- $\exists$ a modified version : path lattice cohomology
- $\exists$ an improvement of $\mathbb{H}^0$: graded roots

Analytic counterpart.... ?

The analytic type 'makes a choice of a part' of $\mathbb{H}^*$...

E.g., at p_g level, some possible choices made by p_g are

- $\mathfrak{sw}(M)$= normalised Euler characteristic of $\mathbb{H}^*$ (splice quotient)
- normalised rank of reduced $\mathbb{H}^0$ (superisolated germs)
- normalised Euler characteristic of path lattice cohomology

(Newton non–degenerate case)

Problem: List all the choices that the analytic structure can make! (for p_g, for $\mathscr{P}(\mathbf{t})$, for .....)

Thank you!