Positivity and algebraic integrability of holomorphic foliations

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The origins

Algebraic differential equations on $\mathbb{C}^2$

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Central problem

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$$\frac{dy}{dx} = \frac{y}{x} \qquad\qquad \frac{dy}{dx} = y$$

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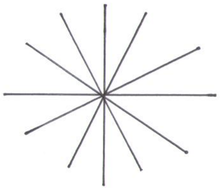
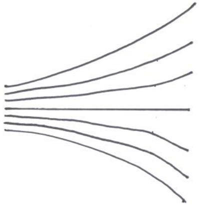


$$y = cx$$

$$\frac{dy}{dx} = y$$



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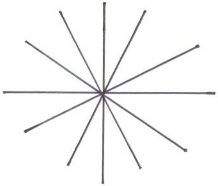

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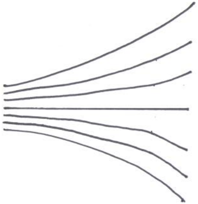

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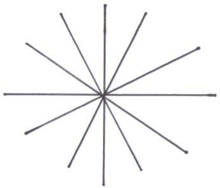

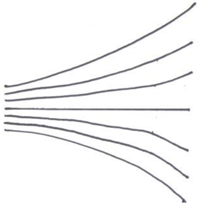

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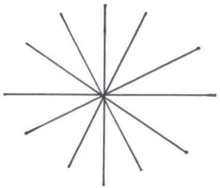

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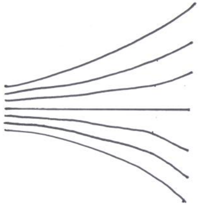

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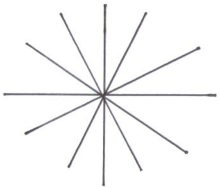

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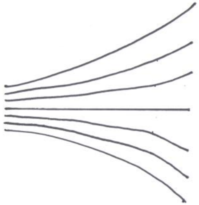

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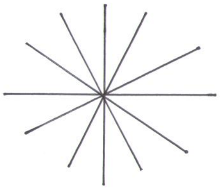

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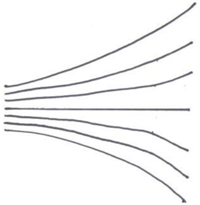

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Holomorphic foliations

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X complex projective manifold

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Definition (Foliation)

Saturated nonzero coherent subsheaf $\mathcal{F} \subsetneq T_X$ satisfying

$$[\mathcal{F}, \mathcal{F}] \subset \mathcal{F} \qquad \text{(Frobenius integrability condition)}$$

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$\varphi : \ X \dashrightarrow Y$ dominant rational map with connected fibers $\rightsquigarrow \ \mathcal{F} \subset T_X$

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Example (Algebraically integrable foliation)

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Algebraic integrability criteria

Theorem (Miyaoka's criterion of uniruledness 1987)

Exactly one of the following holds:

- X is uniruled
- For a general complete intersection curve C on X , $(\Omega^1_X)_{|C}$ is nef

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- Further developments : Bost 2001, Bogomolov and McQuillan 2001, Campana and Paŭn 2015

The Harder-Narasimhan filtration

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- $\exists$ filtration

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_k = \mathcal{E}$$

with $\mathcal{Q}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ μ-semistable, and

$$\mu(\mathcal{Q}_1) > \mu(\mathcal{Q}_2) > \cdots > \mu(\mathcal{Q}_k)$$

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$X^{(n)}$, $\alpha \in \mathrm{Mov}_1(X)$

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$\mathcal{F}$ foliation on complex projective manifold X

Definition

- The **canonical class** of $\mathcal{F}$ is

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- The **index of** $\mathcal{F}$ is

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$$\mathcal{F} \quad \rightsquigarrow \quad 0 \neq \omega \in H^0\big(X, \Omega^{n-r}(-K_X - i(\mathcal{F})A)\big)$$

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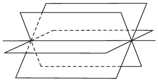

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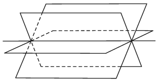

- (Loray-Pereira-Touzet 2011) $i(\mathcal{F}) = r - 1 \implies$ there are 2 types:
  - $\mathcal{F}$ is induced by $\mathbb{P}^n \dashrightarrow \mathbb{P}(2, 1^{n-r})$
  - $\exists\ \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r+1}$ and such that $\mathcal{F} = \varphi^{-1}\mathcal{C}$ for $\mathcal{O} \cong \mathcal{C} \subset T_{\mathbb{P}^{n-r+1}}$

Fano foliations

$\mathcal{F}$ Fano foliation of index $i(\mathcal{F})$ on X

Theorem (A.- Druel - Kovács 2008)

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$\mathcal{F}$ is a **del Pezzo foliation** if $i(\mathcal{F}) = r - 1$

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Classification of del Pezzo foliations

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Classification of del Pezzo foliations

Example (Classification of codimension 1 del Pezzo foliations)

- If $\rho(X) = 1$, then $X \cong \mathbb{P}^n$ or $X \cong Q^n \subset \mathbb{P}^{n+1}$
- If $\rho(X) \geq 2$, then $X = \mathbb{P}(\mathcal{E})$ is a $\mathbb{P}^k$ bundle over $\mathbb{P}^1$ + Classification of $\mathcal{E}$ and $\mathcal{F}$

Classification of del Pezzo foliations high index

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  - Classification of algebraic leaves of del Pezzo foliations
  - **Canonical** and **log canonical** sigularities for foliations (McQuillan 2008)
  - If $\mathcal{F}$ is an algebraically integrable Fano foliation with log canonical singularities, then there is a common point in the closure of every leaf

Stability of T_X for Fano manifolds with $\rho(X) = 1$

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If X is a Fano manifold with $\rho(X) = 1$, then T_X is stable

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If X is a Fano manifold with $\rho(X) = 1$, then T_X is stable

- Suppose T_X is not stable
- $\mathcal{F} \subsetneq T_X$ maximal destabilizing subsheaf

$$\mu(\mathcal{F}) = \frac{i(\mathcal{F})}{rk(\mathcal{F})} > \frac{i(T_X)}{\dim(X)} = \mu(T_X)$$

- $\mathcal{F}$ is a Fano foliation on X

Classification of del Pezzo foliations high index

Theorem

If $\mathcal{F}$ is a del Pezzo foliation on X, then

- either $\mathcal{F}$ is algebraically integrable, or
- $X \cong \mathbb{P}^n$ and $\exists\ \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r+1}$ and such that $\mathcal{F} = \varphi^{-1}\mathcal{C}$ for $\mathcal{O} \cong \mathcal{C} \subset T_{\mathbb{P}^{n-r+1}}$

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$\varphi : X \dashrightarrow Y$ dominant rational map with connected fibers

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Definition

$rk^{alg}(\mathcal{F}) := \dim(X) - \dim(Y)$

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Theorem (A.-Druel 2017)

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- $rk^{alg}(\mathcal{F}) \;=\; i(\mathcal{F}) \;\Longrightarrow\; X \cong \mathbb{P}^n$

  $\exists\ \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-rk^{alg}(\mathcal{F})}$ and $\mathcal{G}$ foliation such that

$$K_{\mathcal{G}} \equiv 0 \quad \text{and} \quad \mathcal{F} = \varphi^*\mathcal{G}$$

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Definition

$\widehat{\imath}(\mathcal{F}) := \max\left\{t \in \mathbb{R} \mid -K_{\mathcal{F}} \equiv tA + E \text{ with } A \text{ ample and } E \in \overline{\mathit{Eff}}^1(X)\right\}$

Bounding the algebraic rank

Theorem (A.-Druel 2017)

$\mathcal{F}$ foliation with $-K_{\mathcal{F}}$ big

- $rk^{alg}(\mathcal{F}) \geq \widehat{\iota}(\mathcal{F})$
- $rk^{alg}(\mathcal{F}) = \widehat{\iota}(\mathcal{F}) \implies X \cong \mathbb{P}^n$

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Definition

$\widehat{\iota}(\mathcal{F}) := \max\left\{t \in \mathbb{R} \mid -K_{\mathcal{F}} \equiv tA + E \text{ with } A \text{ ample and } E \in \overline{Eff}^1(X)\right\}$

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- Cone theorem for rank 2 foliations on 3-folds (Spicer 2017)

Thank you!

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