From continuous rational to regulous functions

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Cartan's umbrella

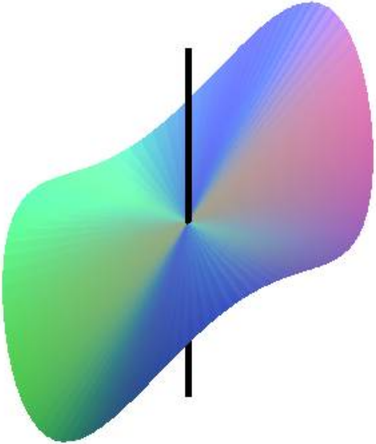

Figure: $\{x^3 - z(x^2 + y^2) = 0\}$

Content and motivations

- Searching for real analogs of the complex utopia
  - arc-symmetric sets, arc-analytic functions -
- Real phenomena not existing in the complex utopia
  - continuous rational $\simeq$ regulous functions -

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Arc-symmetric sets

A subset $E \subset \mathbb{R}^n$ is arc-symmetric if for every analytic arc $\gamma\colon (-1,1) \to \mathbb{R}^n$, with $\gamma((-1,0)) \subset E$, we have $\gamma((-1,1)) \subset E$. In most cases E will be assumed semialgebraic. Algebraic sets are arc-symmetric.

Theorem (Kurdyka 1988)

The semialgebraic arc-symmetric subsets of $\mathbb{R}^n$ are precisely the closed sets of a Noetherian topology $\mathcal{AR}$ on $\mathbb{R}^n$, finer than the Zariski topology.

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Example

The "cloth" of the Cartan umbrella

$$E = \{x^3 - z(x^2 + y^2) = 0,\ x^2 + y^2 \neq 0\} \cup \{(0,0,0)\}$$

is $\mathcal{AR}$-closed, but it is not analytic at the origin of $\mathbb{R}^3$.

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Irreducible arc-symmetric sets

Theorem (Kurdyka 1988)

Assume that E is an $\mathcal{AR}$-closed irreducible subset of $\mathbb{R}^n$, let X be a real algebraic set such that $E \subset X$ and $\dim E = \dim X$. If

$$\pi \colon \tilde{X} \to X$$

is a resolution of singularities of X, then there exists a unique connected component $\tilde{E}$ of $\tilde{X}$ such that $\pi(\tilde{E})$ is the Euclidean closure of the regular part of E. Of course $\pi(\tilde{E}) \subset E$ but in general $\pi(\tilde{E}) \neq E$.

Nash sheets

Definition (Nash 1952)

Let $X \subset \mathbb{R}^n$ be a real algebraic (or $\mathcal{AR}$-closed) set. A subset $S \subset X$ is a sheet of X if it is:

- connected by analytic arcs,
- maximal for this property,
- and has nonempty interior in X.

Theorem (Kurdyka 1988 answering Nash 1952)

Let X be an algebraic subset or $\mathcal{AR}$-closed subset of $\mathbb{R}^n$. Then:

1. *There are finitely many sheets in X.*
2. *Each sheet in X is semialgebraic and Euclidean closed.*
3. *X is the union of its sheets.*

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Arc-analytic functions

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Let $X \subset \mathbb{R}^n$ be $\mathcal{AR}$-closed. A map $f\colon X \to \mathbb{R}^p$ is arc-analytic (a-a) if for every analytic arc $\gamma\colon (-1,1) \to X \subset \mathbb{R}^n$ the composite $f \circ \gamma$ is analytic.

$$f_k(x,y) = \frac{x^{3+k}}{x^2+y^2} \text{ for } (x,y) \neq (0,0) \text{ and } f(0,0) = 0.$$

is a-a and C^k, but not C^{k+1}; $g(x,y) = (x^4+y^4)^{1/2}$ is a-a, but not rational.

1. Every semialgebraic arc-analytic function is Euclidean continuous.
2. $\mathcal{A}_a(X) =$ ring of s-alg. a-a functions on X is not Noetherian (if $\dim X \geq 2$), but any ascending chain of prime ideals of $\mathcal{A}_a(X)$ is stationary.
3. Nullstellensatz holds in $\mathcal{A}_a(X)$, as in the complex case.

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Blow-analytic and arc-analytic functions

Let X be a real analytic manifold, a function $f : X \to \mathbb{R}$ is blow-analytic (blow Nash) in the sense of Kuo iff there exists a $\pi : X' \to X$, which is the composite of a finite sequence of blowups with smooth Zariski closed centers, such that $f \circ \pi : X' \to \mathbb{R}$ is an analytic (Nash) function.

Theorem (Bierstone and Milman 1991)

Let X be a smooth real algebraic set. A semialgebraic function $f : X \to \mathbb{R}$ is arc-analytic iff it is blow-Nash.

Conjecture Let X be an analytic manifold. Assume that $f : X \to \mathbb{R}$ is an arc-analytic function with subanalytic graph. Then f is blow-analytic. A weak version of the conjecture was established by BM. (The converse is true.)

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Regular and rational functions

Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f\colon W \to \mathbb{R}$ be a function defined on $W \subset X$:

- f is regular at a point $x \in W$ if there exist two polynomial functions p, q on $\mathbb{R}^n$ such that $q(x) \neq 0$ and $f = p/q$ on $W \cap \{q \neq 0\}$.
- f is a regular function if it is regular at each point of W.
- Any rational function R on X determines a regular function $R\colon X \setminus \mathrm{Pole}(R) \to \mathbb{R}$, where $\mathrm{Pole}(R)$ stands for the polar set of R.

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Regulous functions 1

Let X be a real algebraic set, $f\colon W \to \mathbb{R}$ a function defined on some subset $W \subset X$, and Y the Zariski closure of W in X. A rational function R on Y is said to be a rational representation of f if there exists a Zariski open dense subset $Y^0 \subset Y \setminus \mathrm{Pole}(R)$ such that $f|_{W\cap Y^0} = R|_{W\cap Y^0}$.

The following conditions are equivalent:

1. $f|_{W\cap Z}$ has a rational representation for every algebraic subset $Z \subset X$.
2. There exists a sequence of algebraic subsets

$$X = X_0 \supset X_1 \supset \cdots \supset X_{m+1} = \varnothing$$

such that the restriction of f is regular on $W \cap (X_i \setminus X_{i+1})$ for all i.

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Regulous functions 2

Example (Kollár)

Let $S := (x^3 - (1+z^2)y^3 = 0) \subset \mathbb{R}^3$ and let $f : S \to \mathbb{R}$,

$$f(x,y,z) = (1+z^2)^{1/3}.$$

Note that $\mathrm{Sing}(S) = (z\text{-axis})$ and $f(x,y,z) = x/y$ on $S \setminus (z\text{-axis})$; f is continuous and has a rational representation but f is not regulous since $f|_{z\text{-axis}}$ does not have a rational representation.

Theorem (Kollár and Nowak 2015)

Let X be a smooth real algebraic set and let $f : W \to \mathbb{R}$ be a function defined on an algebraic subset $W \subset X$. Then the following conditions are equivalent:

1. *f is regulous.*
2. *$f = \tilde{f}|_W$, where $\tilde{f} : X \to \mathbb{R}$ is continuous with rational representation.*

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Theorem

Let X be a smooth real algebraic set. A function $f : X \to \mathbb{R}$ is regulous iff there exists a $\pi : X' \to X$, which is the composite of a finite sequence of blowups with smooth Zariski closed centers, such that $f \circ \pi : X' \to \mathbb{R}$ is a regular function.

In particular, regulous functions on smooth sets are arc-analytic and semi-algebraic.

Curve-regulous

Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f\colon W \to \mathbb{R}$ be a function defined on $W \subset X$:

- f is curve-regulous if for every irreducible algebraic curve $C \subset X$ the restriction $f|_{W \cap C}$ is regulous (equivalently, $f|_{W \cap C}$ is continuous and has a rational representation).
- f is arc-regulous if for every irreducible algebraic curve $C \subset X$ and every point $x \in W \cap C$ there exists an open neighborhood $U_x \subset W$ of x such that the restriction $f|_{U_x \cap C}$ is regulous .

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Curve-regulous are actually regulous

Theorem (Kollár, Kucharz and Kurdyka 2017)

Let X be a real algebraic set and let $W \subset X$ be a subset that is either open or semialgebraic. A function $f : W \to \mathbb{R}$ is regulous iff it is curve-regulous.

The corresponding result for arc-regulous functions takes the following form.

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Constructible topology and k-regulous functions 1

- $\mathcal{R}^k(U) =$ ring of k-regulous functions (C^k and regulous), $k \geq 0$, on U open in $\mathbb{R}^n$;
- $f_k = \frac{x^{3+k}}{x^2+y^2}$ is such a function;
- $\mathcal{R}^k(U) \subset \mathcal{A}_a(U)$, i.e., any regulous function is arc-analytic and s-alg., also $\mathcal{R}^k(U)$ is not Noetherian (if $n \geq 2$);
- since ∞-regulous $=$ regular, we consider only k finite.

A subset $A \subset \mathbb{R}^n$ is constructible if it belongs to the Boolean algebra generated by the algebraic subsets of $\mathbb{R}^n$. The Euclidean closed constructible subsets of $\mathbb{R}^n$ are precisely the closed sets of a Noetherian topology $\mathcal{C}$ on $\mathbb{R}^n$. We have $\{$*Zariski topology*$\} \subset \{\mathcal{C}$ *topology*$\} \subset \{\mathcal{AR}$ *topology*$\}$.

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Constructible topology and k-regulous functions 2

Theorem (Fichou, Huisman, Mangolte and Monnier 2015)

For a subset $E \subset \mathbb{R}^n$, the following conditions are equivalent:

1. $E = Z(I)$ *for some ideal* I *of* $\mathcal{R}^k(\mathbb{R}^n)$.
2. $E = Z(f)$ *for some function* f *in* $\mathcal{R}^k(\mathbb{R}^n)$.
3. E *is Euclidean closed and constructible (i.e.,* $\mathcal{C}$*-closed).*

Regulous Nullstellensatz

The classical Nullstellensatz fails for regular functions over $\mathbb{R}$, but

Theorem (Fichou, Huisman, Mangolte and Monnier 2015)

Let I be an ideal of the ring $\mathcal{R}^k(\mathbb{R}^n)$. If a function f in $\mathcal{R}^k(\mathbb{R}^n)$ vanishes on $Z(I)$, then f^m belongs to I for some positive integer m.

Example

Let $I = (g)$, where $g(x, y) = x^2 + y^2$. Clearly $h(x, y) = x$ vanishes on $Z(I) = \{(0, 0)\}$. Then

$$h^{3+k} = x^{3+k} = gf_k \in I,$$

since $f_k(x, y) = \frac{x^{3+k}}{x^2+y^2} \in \mathcal{R}^k(\mathbb{R}^n)$.

Cartan's theorems A and B

The assignment $\mathcal{R}^k\colon U \mapsto \mathcal{R}^k(U)$, where U is a $\mathcal{C}$-open subset of $\mathbb{R}^n$, is a sheaf of rings on $\mathbb{R}^n$. Sheaves of $\mathcal{R}^k$-modules on $\mathbb{R}^n$ are called k-regulous sheaves.

It follows from the above theorem that Cartan's theorems A and B are available for k-regulous sheaves.

Theorem (Fichou, Huisman, Mangolte and Monnier 2015)

If $\mathcal{F}$ is a quasi-coherent k-regulous sheaf on $\mathbb{R}^n$, then:

1. *$\mathcal{F}$ is generated by global sections.*
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Approximation by regular maps

Conjecture 1. Let X be a compact real algebraic set. For a continuous map $f: X \to \mathbb{S}^p$, the following are equivalent:

1. f can be approximated by regular maps.
2. f is homotopic to a regular map.

Conjecture 2. For any pair (n, p) of positive integers, each continuous map $\mathbb{S}^n \to \mathbb{S}^p$ can be approximated by regular maps.

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Example (Bochnak and Kucharz 1993) A continuous map $(\mathbb{S}^1)^n \to \mathbb{S}^2$ can be approximated by regular maps iff it is null homotopic.

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Example (Bochnak and Kucharz 1993) A continuous map $(\mathbb{S}^1)^n \to \mathbb{S}^2$ can be approximated by regular maps iff it is null homotopic.

Approximation by regulus maps

Theorem (Kucharz 2009, 2014)

Let X be a compact smooth real algebraic set. If $f : X \to \mathbb{S}^p$ is a continuous map, $\dim X = p$, then:

1. *f is homotopic to a k-regulous map, for any integer $k \geq 0$.*
2. *f can be approximated by regulous maps.*

In general this theorem does not hold if $\dim X > p$.

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2. *f can be approximated by regulous maps.*

Algebraic vector bundles

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). Which topological $\mathbb{F}$-vector bundles on a real algebraic set X admit an algebraic (regulous) structure?

Theorem (Fossum 1969, Swan 1977)

Any topological $\mathbb{F}$-vector bundle on $\mathbb{S}^n$, $n \geq 1$, admits an algebraic structure.

Theorem (Bochnak, Buchner and Kucharz 1989)

For any integer $d \geq 1$, there exists a smooth real algebraic set $\Sigma^{4d} \subset \mathbb{R}^{4d+1}$, ε-isotopic to $\mathbb{S}^{4d}$, such that the following holds:
A topological $\mathbb{F}$-vector bundle on Σ^{4d} admits an algebraic structure iff it is stably trivial.

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Regulus vector bundles

Theorem (Kucharz and Kurdyka 2016)

Let X be a compact real algebraic set. If X is homemorphic to $\mathbb{S}^n$, then each topological $\mathbb{F}$-vector bundle on X admits a regulous structure.

This is in sharp contrast with the algebraic case, cf. the previous theorem.

Theorem (Kucharz and Kurdyka 2016)

Let X be a compact real algebraic set. A topological $\mathbb{F}$-vector bundle ξ on X admits a regulous structure if and only if the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}$ (obtained from ξ by restricting scalars to $\mathbb{R}$) admits a regulous structure.

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Thank you for your attention