# Interaction between singularity theory and the minimal model program

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ICM, Rio, August 3rd 2018

- Given a singularity $x \in X$ in algebraic geometry, the first thing people would try is to resolve it. That is to find a morphism $f\colon Y \to X$, with $Y$ being smooth, and there exists a lower dimensional subspace $Z \subset X$ such that $f$ is isomorphic over $X \setminus Z$. Such $X$ and $Y$ are called birational.
- In characteristic 0, this is always possible by Hironaka's Resolution of Singularity. However, when dimension is at least three, there is often no 'best' (i.e. minimal) choice of $Y$.
- There is an alternative way to produce birational models, via the minimal model program. That is given any smooth variety $X$, we produce a sequence of birational models by an algorithm: $X = X_0, X_1, ...$, until stop with a model with more canonical properties. Such a model is called a minimal model.

- (Italian school) It works perfectly well, in dimension 2.
- (Mori, Reid) In 80s, one important observation: when dimension is at least three, if the minimal model program can be run, singular varieties appear.
- Nevertheless, the singularities appearing are not random, but controllable by some invariants. It forms a class of singularities which we need to understand, for a better knowledge of the minimal model program.
- Later, it turns out this class of singularities also naturally arise in many other contexts, notably Kähler-Einstein metric, Frobenius action, Mirror symmetry etc.
- The development of singularity theory and global theory in higher dimensional geometry intertwine.

- Theme 1 Given a singularity $x \in X$, study the topology of

$$\mathrm{Link}(x \in X) = (X \setminus \{x\}) \cap B_\epsilon(x, \mathbb{C}^N),$$

  and its non-archimedean analogy.
- Theme 2 For any holomorphic function $f$, consider

$$c = \limsup\{c \mid \frac{1}{|f|^{2c}} \text{ is locally integrable}\}.$$

  Study properties of the set $\{c| \dim(X) = n \text{ for a fixed } n\}$. Shokurov conjectured it satisfies the ascending chain condition (ACC).
- Theme 3 Is there a stability notion of a singularity? We want to develop a 'K-stability theory' for a large class of singularities.

- Given a normal space $X$, and $\mathbb{Q}$-Gorenstein, (i.e. the class given by $K_X$ is torsion in the class group). Let $f\colon Y \to X$ be a resolution, such that $\mathrm{Ex}(f)$ is simple normal crossing. Write $K_{Y/X} = \sum_i a_i(X; E_i)E_i$, then we call $X$ is

$$\begin{cases} \text{terimnal} & \text{if } a_i(X; E_i) > 0; \\ \text{canonical} & \text{if } a_i(X; E_i) \geq 0; \\ \text{Kawamata log terminal (klt)} & \text{if } a_i(X; E_i) > -1; \\ \text{log canonical (lc)} & \text{if } a_i(X; E_i) \geq -1. \end{cases}$$

  for all $i$.
- Each class is preserved under the minimal model program.
- Giving an explicit classification is too complicated beyond dimension three.
- Consider a pair $(X, \Delta)$ rather than only $X$, and we can similarly define $a_i(X, \Delta; E_i)$ and the above notions of singularities for pairs.

- Similar to the canonical properties of a minimal model, there is no canonical resolution of a singularity, but a relative MMP can provide birational models with canonical properties.
- More precisely, consider a $\mathbb{Q}$-Gorenstein isolated singularity $x \in X$, MMP yields a model $g\colon X^{\rm dlt} \to X$ such that $\Delta := \mathrm{Ex}(g) = g^{-1}(x) = \sum_i E_i$, and $(X^{\rm dlt}, \Delta)$ is divisorial log terminal, that is simple normal crossing on an open set $U \subset X^{\rm dlt}$, and $a_i(X^{\rm dlt}, \Delta; E) > -1$ for any $\mathrm{Center}_E(X^{\rm dlt}) \subset X^{\rm dlt} \setminus U$. Moreover, $K_{X^{\rm dlt}} + \Delta$ is nef, i.e. $(K_{X^{\rm dlt}} + \Delta) \cdot C \geq 0$ for any $C$ such that $g(C)$ is a point.
- Any two $X_i^{\rm dlt}$ satisfying the above properties, the pull backs of $K_{X_i^{\rm dlt}} + \Delta_i$ are the same on a common model.

- We can define a regular cell complex $\mathcal{D}(\Delta)$ to characterize how $E_i$ intersect each other: for each $E_i$, we put a vertex $v_i$; for each component $E_i \cap E_j$, we put an edge $v_{ij}$; for each component $E_i \cap E_j \cap E_k$, we put a two dimensional face $v_{ijk}$; and so on. So $\dim \mathcal{D}(\Delta) \leq \dim(X) - 1$.
- (de Fernex-Kollár-X.) $\mathcal{D}(\Delta)$ does not depend on the choice of $X^{\text{dlt}}$. We denote it by $\mathcal{DMR}(x \in X)$.
- (Berkovich, Thuillier) If we take a log resolution $(Y, \mathrm{Ex}(f)) \to X$ and form $\mathcal{D}(\mathrm{Ex}(f))$, then the non-archimedean link $\mathrm{Link}^{\mathrm{NA}}(x \in X)$ has a strong deformation retract to $\mathcal{D}(\mathrm{Ex}(f))$.

**Theorem (de Fernex-Kollár-X.)**

$\mathcal{D}(\mathrm{Ex}(f))$ admits a strong deformation retract to $\mathcal{DMR}(x \in X)$.

- We can generalize the above setting to a degeneration of a smooth minimal variety $X/\mathrm{Spec}\,\mathbb{C}((t))$ by taking the dual complex of the special fiber of a minimal dlt model $\mathfrak{X}/\mathrm{Spec}\,\mathbb{C}[[t]]$.
- (Nicaise-X.) For the degeneration of a Calabi-Yau manifold $X$, our definition coincides with the essential skeleton $\mathrm{Sk}(X)$ defined by Kontsevich-Soibelman, which is the base of the non-archimedean SYZ fibration.
- MMP gives a good tool to study $\mathrm{Sk}(X)$.

**Theorem (Kollár-X.)**

When $X/\mathrm{Spec}\,\mathbb{C}((t))$ is a strict CY of $\dim(X) \le 3$ with the maximal degeneration type. Assume it admits a reduced minimal dlt model, then $\mathrm{Sk}(X) \simeq \mathbb{S}^3$.

- Given $X$, $c = \limsup\{c \mid \frac{1}{|f|^{2c}}$ is locally integrable$\}$ is the same as

$$\mathrm{lct}(X; V(f)) := \limsup\{c \mid (X, c \cdot V(f)) \text{ is log canonical.}\}$$

- We prove the following Shokurov's conjecture:

**Theorem (Hacon-McKernan-X.)**

The set $\mathrm{LCT}_n = \{\mathrm{lct}(X, (f)) | \dim(\mathrm{X}) = n\}$ satisfies ACC.

- (de Fernex-Ein-Mustaţă) When $X = \mathbb{C}^n$.
- Our proof connects it to global results.

**Theorem (Global ACC Conjecture, Hacon-McKernan-X.)**

If $I$ is a set satisfying the descending chain condition (DCC). There exists a finite set $I_0$ such that if $(X, \Delta)$ is an $n$-dimensional lc pair with $K_X + \Delta \equiv 0$ and the coefficients of $\Delta$ are in $I$. Then the coefficients of $\Delta$ are in $I_0$.

- In fact, it is a general local-to-global principle that there is a correspondence:

$$\begin{aligned} \text{klt singularities} &\quad\longleftrightarrow\quad \text{Fano} \\ \text{strict lc singularities} &\quad\longleftrightarrow\quad \text{Calabi-Yau} \end{aligned}$$

- Let $E$ over $X$ such that $a(E) = -1$. Then we can find a model $g : Z \to X$ such that

$$g^*(K_X + c \cdot V(f)) = K_Z + c \cdot g_*^{-1}(V(f)) + E.$$

Restricting on $E$, we get a log Calabi-Yau pair.

- We can get the global ACC conjecture from the following uniform result for log general type pairs:

**Theorem (Alexeev-Kollár Conjecture, Hacon-McKernan-X.)**

If $I \subset [0,1]$ is a set satisfying the descending chain condition (DCC). Then

1. $\mathcal{V}(I,n) := \{\mathrm{vol}(K_X+\Delta) |$ where $\dim(X)=n$, coefficients of $\Delta$ are in $I$, $(X,\Delta)$ is log canonical$\}$ form a DCC set. In particular, $\mathcal{V}(I,n)\cap(0,\infty)$ has a minimum.
2. there exists $N=N(I,n)$, such that if $K_X+\Delta$ is big, then $|N(K_X+\Delta)|$ is birational.

- The above is a key point to show the KSBA moduli space, which is the moduli space compactifying the one parametrizing canonically polarized manifolds, is bounded.
- Effective results in general dimension are not known.

- We look at the correspondence between Fano varieties and klt singularities.
- K-stability questions on Fano varieties have inspired a lot of new higher dimensional geometry questions.
- Goal: develop a stability theory of singularities similar to the K-stability of Fano varieties.
- Given a klt singularity, $x \in X = \mathrm{Spec}(R)$. Consider the space of valuations $\mathrm{Val}_{X,x} = \{\text{Valuations on } K(X) \text{ whose center is } x\}$.
- Define the log discrepancy function $A_X$ on $\mathrm{Val}_{X,x}$ extending $A(\mathrm{ord}_{E_i}) = a_i(E_i) + 1$.
- (Chi Li) Define $\widehat{\mathrm{vol}}_{X,x}(v) := \mathrm{vol}(v) \cdot A_X^n(v)$ on $\mathrm{Val}_{X,x}$, where $\mathrm{vol}(v) = \lim_{k\to\infty} \frac{\mathrm{length}(R/\mathfrak{a}_k)}{k^n/n!}$ and $\mathfrak{a}_k = \{f \mid v(f) \geq k\}$.
- The minimizer $v$ of $\widehat{\mathrm{vol}}$ carries a deep geometric information.

**Conjecture (Stable Degeneration Conjecture, Li, Li-X.)**

Given any klt singularities $x \in X$, up to a rescaling, there is a unique (up to scaling) minimizer $v \in \mathrm{Val}_{X,x}$ of $\widehat{\mathrm{vol}}_{X,x}$, which is quasi-monomial, with a finitely generated associated graded ring $R_0 = \mathrm{gr}_v(R) := \oplus_{k\in\Phi}(\mathfrak{a}_k/\mathfrak{a}_{>k})$, and $(X_0 = \mathrm{Spec}(R_0), \lambda_v)$ is a klt K-semistable Fano cone.

- the grading yields a $T = (\mathbb{C}^*)^r$-action ($r$ is the $\mathbb{Q}$-rank of $v$, i.e. the rank of the $G(\Phi) = \mathbb{Z}^r$) on $X_0$, where $v$ induces a Reeb vector $\lambda_v$.
- (Collins-Székelyhidi) Define K-semistability of a Fano cone.
- A quasi-monomial valuation with a finitely generated associated graded ring is called a K-semistable valuation if $(X_0 = \mathrm{Spec}(R_0), \lambda_v)$ is a klt K-semistable Fano cone.
- Inspired by the metric tangent cone construction in differential geometry.

- Known results on SDC conjecture:
  (Blum) Existence;
  (Li, Li-Liu, Li-X.) K-semistability implies minimizing;
  (Li-X.) Uniqueness among K-semistable valuations (answer a conjecture of Donaldson-Sun);
  etc..
- The missing part is that a minimizer is quasi-monomial, and the associated graded ring is finitely generated (known for the divisorial case).

- Recently, after using valuations to give an equivalent definition of K-(semi,poly)stability of Fano varieties, there has been a lot of progress.
- Its combination with birational geometry gives a major step forward.
- Two guiding questions:
  Question 1: How to check an example of a Fano variety is K-(semi,poly)stable?
  Question 2: Using K-(semi,poly)stability to construct a projective moduli space of K-polystable Fano varieties.
- There has been a lot of progress in both questions.

Thank you very much!