Recursive combinatorial aspects of compactified moduli spaces

Lucia Caporaso
Department of Mathematics and Physics,
Roma Tre University - Rome, Italy

ICM 2018 - 02/08/2018

The program in general

A moduli space, M, has a natural algebraic structure, not complete (i.e. degenerations must occur).

Step 1: A completion, $\overline{M}$, of M has a recursive structure by topological/combinatorial type.

The program in general

A moduli space, M, has a natural algebraic structure, not complete (i.e. degenerations must occur).

Step 1: A completion, $\overline{M}$, of M has a recursive structure by topological/combinatorial type.

Step 2: There is a “cone complex", $\overline{\Sigma}(\overline{M})$, associated to this structure which is also a moduli space for combinatorial objects.

The program in general

A moduli space, M, has a natural algebraic structure, not complete (i.e. degenerations must occur).

Step 1: A completion, $\overline{M}$, of M has a recursive structure by topological/combinatorial type.

Step 2: There is a "cone complex", $\overline{\Sigma}(\overline{M})$, associated to this structure which is also a moduli space for combinatorial objects.

Step 3: $\overline{\Sigma}(\overline{M})$ is also the skeleton of the analytification, $\overline{M}^{an}$, of $\overline{M}$.

The program in general

A moduli space, M, has a natural algebraic structure, not complete (i.e. degenerations must occur).

Step 1: A completion, $\overline{M}$, of M has a recursive structure by topological/combinatorial type.

Step 2: There is a "cone complex", $\overline{\Sigma}(\overline{M})$, associated to this structure which is also a moduli space for combinatorial objects.

Step 3: $\overline{\Sigma}(\overline{M})$ is also the skeleton of the analytification, $\overline{M}^{\text{an}}$, of $\overline{M}$.

The retraction $\rho : \overline{M}^{\text{an}} \longrightarrow \overline{\Sigma}(\overline{M})$ has a remarkable geometric interpretation in terms of the all moduli spaces involved.

Program applies to moduli of smooth/stable curves

$M = \mathcal{M}_{g,n}$ moduli stack of smooth n-pointed curves of genus g.
$\overline{\mathcal{M}}_{g,n}$ moduli stack of *stable n-pointed curves* of genus g.

[Deligne-Mumford, Knudsen]

Program applies to moduli of smooth/stable curves

$M = \mathcal{M}_{g,n}$ moduli stack of smooth n-pointed curves of genus g.
$\overline{\mathcal{M}}_{g,n}$ moduli stack of *stable n-pointed curves* of genus g.

[Deligne-Mumford, Knudsen]

$\mathcal{G}_{g,n}$ graded poset of *stable graphs of genus g with n legs*.

$$\sigma_M : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{G}_{g,n}; \qquad X \mapsto G \text{ dual graph (topological type) of } X$$

Program applies to moduli of smooth/stable curves

$M = \mathcal{M}_{g,n}$ moduli stack of smooth n-pointed curves of genus g.
$\overline{\mathcal{M}}_{g,n}$ moduli stack of *stable n-pointed curves* of genus g.

[Deligne-Mumford, Knudsen]

$\mathcal{G}_{g,n}$ graded poset of *stable graphs of genus g with n legs*.

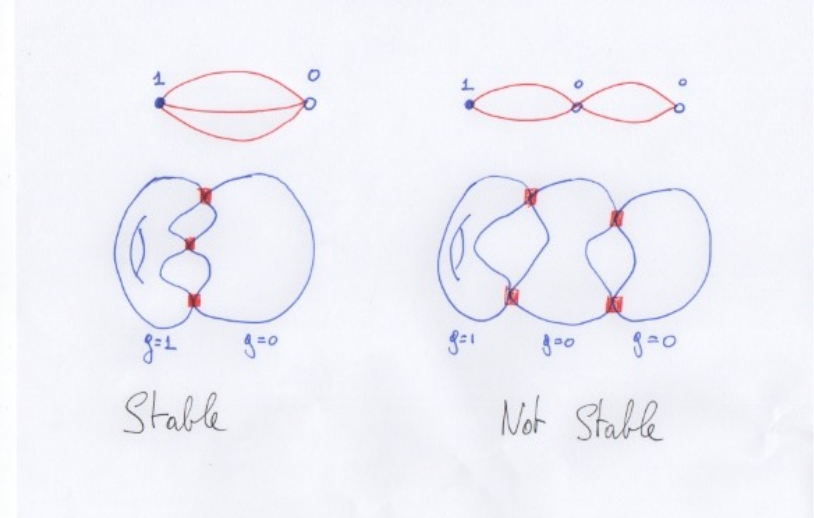
$$\sigma_M : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{G}_{g,n}; \qquad X \mapsto G \text{ dual graph (topological type) of } X$$

The fibers of σ_M form a recursive stratification of $\overline{\mathcal{M}}_{g,n}$:

$$\overline{\mathcal{M}}_{g,n} = \sqcup_{G \in \mathcal{G}_{g,n}} M_G$$

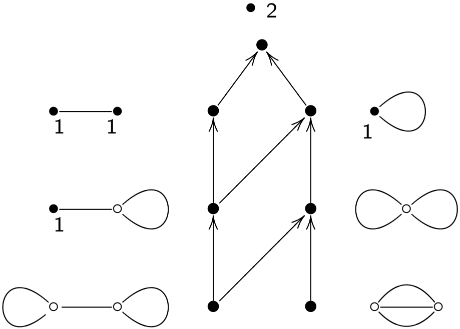
$M_G := \sigma_M^{-1}(G)$.

Curves over $\mathbb{C}$ and their dual graph: $g = 3$, $n = 0$



Stable graphs of genus 2 with no legs

The graded poset $\mathcal{G}_{2,0}$



Program applies to moduli of smooth/stable curves

$\overline{\mathcal{M}}_{g,n}$ moduli stack of stable (n-pointed, genus g) curves.
$\mathcal{G}_{g,n}$ stable graphs (of genus g with n legs).

$$\sigma_M : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{G}_{g,n}; \qquad X \mapsto G \text{ dual graph (topological type) of } X$$

$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ is the colimit of a cone complex on the graded poset $\mathcal{G}_{g,n}$.

Program applies to moduli of smooth/stable curves

$\overline{\mathcal{M}}_{g,n}$ moduli stack of stable (n-pointed, genus g) curves.
$\mathcal{G}_{g,n}$ stable graphs (of genus g with n legs).

$$\sigma_M : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{G}_{g,n}; \qquad X \mapsto G \text{ dual graph (topological type) of } X$$

$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ is the colimit of a cone complex on the graded poset $\mathcal{G}_{g,n}$.

$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \cong \overline{M}_{g,n}^{\mathrm{trop}}$ moduli space of *(extended) n-pointed tropical curves.* [Mikhalkin, Brannetti-Melo-Viviani, Abramovich-C-Payne]

Program applies to moduli of smooth/stable curves

$\overline{\mathcal{M}}_{g,n}$ moduli stack of stable (n-pointed, genus g) curves.
$\mathcal{G}_{g,n}$ stable graphs (of genus g with n legs).

$$\sigma_M : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{G}_{g,n}; \qquad X \mapsto G \text{ dual graph (topological type) of } X$$

$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ is the colimit of a cone complex on the graded poset $\mathcal{G}_{g,n}$.

$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \cong \overline{M}_{g,n}^{\mathrm{trop}}$ moduli space of *(extended) n-pointed tropical curves*. [Mikhalkin, Brannetti-Melo-Viviani, Abramovich-C-Payne]

$\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ Berkovich analytification of $\overline{\mathcal{M}}_{g,n}$, parametrizes stable curves over non-Archimedean valuation fields up to equivalence.

[Berkovich, Thuillier]

The tropicalization map

A K-point in $\overline{\mathcal{M}}_{g,n}^{\text{an}}$ is a stable curve $\mathcal{X}_K \to \operatorname{Spec} K$ over the non-Archimedean valuation field K, up to equivalence.

The tropicalization map

A K-point in $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ is a stable curve $\mathcal{X}_K \to \operatorname{Spec} K$ over the non-Archimedean valuation field K, up to equivalence.
A point in $\overline{M}_{g,n}^{\mathrm{trop}}$ is a tropical curve (G, ℓ) up to equivalence, where $\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$.

The tropicalization map

A K-point in $\overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$ is a stable curve $\mathcal{X}_K \to \operatorname{Spec} K$ over the non-Archimedean valuation field K, up to equivalence.

A point in $\overline{M}^{\mathrm{trop}}_{g,n}$ is a tropical curve (G, ℓ) up to equivalence, where $\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$.

The *tropicalization map* [Baker-Payne-Rabinoff,Tyomkin, Viviani, Abramovich-C-Payne]

$$\mathrm{trop} : \overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \xrightarrow{\ \rho\ } \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\ \cong\ } \overline{M}^{\mathrm{trop}}_{g,n}$$

$$[\mathcal{X}_K \to \operatorname{Spec} K] \longmapsto [(G, \ell)]$$

The tropicalization map

A K-point in $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ is a stable curve $\mathcal{X}_K \to \operatorname{Spec} K$ over the non-Archimedean valuation field K, up to equivalence.

A point in $\overline{M}_{g,n}^{\mathrm{trop}}$ is a tropical curve (G, ℓ) up to equivalence, where $\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$.

The *tropicalization map* [Baker-Payne-Rabinoff,Tyomkin, Viviani, Abramovich-C-Payne]

$$\begin{array}{ccccc} \mathrm{trop} : \overline{\mathcal{M}}_{g,n}^{\mathrm{an}} & \xrightarrow{\rho} & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) & \xrightarrow{\cong} & \overline{M}_{g,n}^{\mathrm{trop}} \\ [\mathcal{X}_K \to \operatorname{Spec} K] & \longmapsto & & & [(G,\ell)] \end{array}$$

G is the dual graph of the *reduction*, a stable curve X, of $\mathcal{X}_K \to \operatorname{Spec} K$ over $\operatorname{Spec} R$ ($R \subset K$ the valuation ring of K).

The tropicalization map

A K-point in $\overline{\mathcal{M}}^{\text{an}}_{g,n}$ is a stable curve $\mathcal{X}_K \to \operatorname{Spec} K$ over the non-Archimedean valuation field K, up to equivalence.

A point in $\overline{M}^{\text{trop}}_{g,n}$ is a tropical curve (G, ℓ) up to equivalence, where $\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$.

The *tropicalization map* [Baker-Payne-Rabinoff,Tyomkin, Viviani, Abramovich-C-Payne]

$$\begin{array}{ccccc} \text{trop} : \overline{\mathcal{M}}^{\text{an}}_{g,n} & \xrightarrow{\ \rho\ } & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) & \xrightarrow{\ \cong\ } & \overline{M}^{\text{trop}}_{g,n} \\ [\mathcal{X}_K \to \operatorname{Spec} K] & & \longmapsto & & [(G,\ell)] \end{array}$$

G is the dual graph of the *reduction*, a stable curve X, of $\mathcal{X}_K \to \operatorname{Spec} K$ over $\operatorname{Spec} R$ ($R \subset K$ the valuation ring of K).

$\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$ measures, by the valuation of K, the local geometry of the total space at X.

Program applies to Hurwitz space/admissible covers

$\mathcal{H}_\bullet = \mathcal{H}_{g,h}(\underline{\pi})$ = Hurwitz space of degree-d covers of a smooth curve of genus h by a smooth curve of genus g with exactly b branch points and ramification prescribed by $\underline{\pi} = (\pi_1, \ldots, \pi_b)$, with π_i a partition of d. It is not complete

$\overline{\mathcal{H}}_\bullet = \overline{\mathcal{H}}_{g,h}(\underline{\pi})$ moduli stack of *admissible covers*.

[Harris-Mumford, Mochizuchi, Abramovich-Corti-Vistoli]

Program applies to Hurwitz space/admissible covers

$\mathcal{H}_\bullet = \mathcal{H}_{g,h}(\underline{\pi})$ = Hurwitz space of degree-d covers of a smooth curve of genus h by a smooth curve of genus g with exactly b branch points and ramification prescribed by $\underline{\pi} = (\pi_1, \ldots, \pi_b)$, with π_i a partition of d. It is not complete

$\overline{\mathcal{H}}_\bullet = \overline{\mathcal{H}}_{g,h}(\underline{\pi})$ moduli stack of *admissible covers*.

[Harris-Mumford, Mochizuchi, Abramovich-Corti-Vistoli]

There are two canonical morphisms, mapping an admissible cover to its target, or its source (for suitable $m, n \in \mathbb{N}$)

$$\overline{\mathcal{M}}_{h,m} \xleftarrow{\;tgt\;} \overline{\mathcal{H}}_{g,h}(\underline{\pi}) \xrightarrow{\;src\;} \overline{\mathcal{M}}_{g,n}$$

Program applies to Hurwitz space/admissible covers

Hurwitz space $= \mathcal{H}_\bullet \subset \overline{\mathcal{H}}_\bullet =$ moduli stack of admissible covers.

$\mathcal{A}_\bullet$ *admissible covers of graphs*, a graded poset.
The recursive stratification [Cavalieri-Markwig-Ranganathan]:

$$\sigma_H : \overline{\mathcal{H}}_\bullet \longrightarrow \mathcal{A}_\bullet; \qquad [C' \to C] \mapsto [G' \to G] \text{ dual graph cover}$$

Program applies to Hurwitz space/admissible covers

Hurwitz space $= \mathcal{H}_\bullet \subset \overline{\mathcal{H}}_\bullet =$ moduli stack of admissible covers.

$\mathcal{A}_\bullet$ *admissible covers of graphs*, a graded poset.
The recursive stratification [Cavalieri-Markwig-Ranganathan]:

$$\sigma_H : \overline{\mathcal{H}}_\bullet \longrightarrow \mathcal{A}_\bullet; \qquad [C' \to C] \mapsto [G' \to G] \text{ dual graph cover}$$

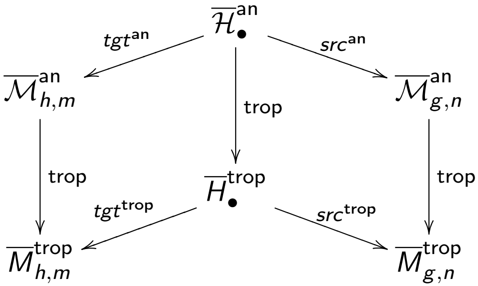
$\overline{\Sigma}(\overline{\mathcal{H}}_\bullet)$ is the colimit of a cone complex on $\mathcal{A}_\bullet$.
$\overline{\Sigma}(\overline{\mathcal{H}}_\bullet) \cong \overline{H}_\bullet^{\text{trop}}$ moduli space of *tropical admissible covers*.

$$\text{trop} : \overline{\mathcal{H}}_\bullet^{\text{an}} \longrightarrow \overline{H}_\bullet^{\text{trop}}$$

Compatible with the program for $\overline{\mathcal{M}}_{g,n}$ via the canonical maps *src* and *tgt*, and their tropical and anlytic counterparts.

Compatibility of tropicalization maps with canonical maps

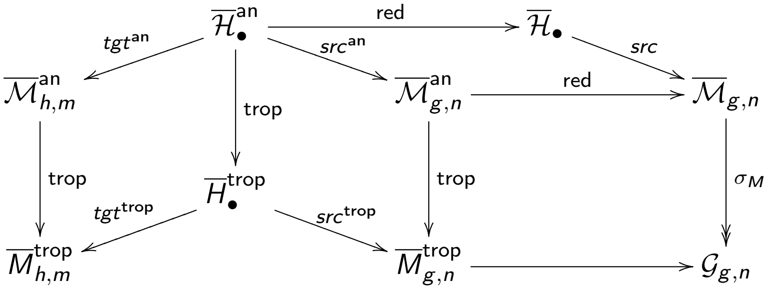
A commutative diagram



[Cavalieri-Markwig-Ranganathan]

Compatibility of tropicalization maps with canonical maps

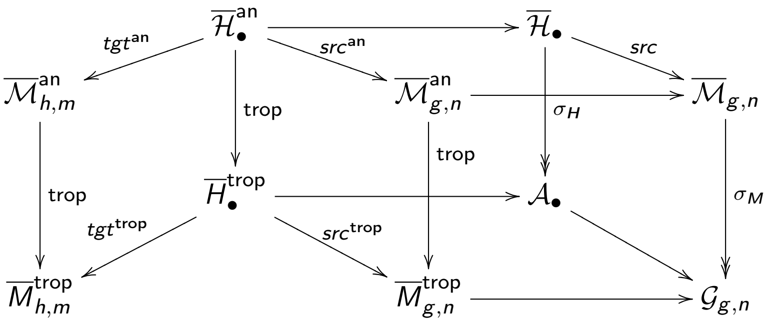
A commutative diagram



[Cavalieri-Markwig-Ranganathan]

Compatibility of tropicalization maps with canonical maps

A commutative diagram



[Cavalieri-Markwig-Ranganathan]

Moduli of line bundles on a stable curve

X stable curve, G its dual graph,
$E(G) =$ nodes of X and $V(G) =$ irreducible components of X.
$\mathrm{Pic}^d(X) = \sqcup_{|\underline{d}|=d} \mathrm{Pic}^{\underline{d}}(X)$ is the moduli space of line bundles of degree d on X.
$\mathrm{Pic}^{\underline{d}}(X) \cong \mathrm{Jac}(X)$ for all $\underline{d} \in \mathbb{Z}^{V(G)}$. It is not complete.

Moduli of line bundles on a stable curve

X stable curve, G its dual graph,
$E(G) =$ nodes of X and $V(G) =$ irreducible components of X.
$\mathrm{Pic}^d(X) = \sqcup_{|\underline{d}|=d} \mathrm{Pic}^{\underline{d}}(X)$ is the moduli space of line bundles of degree d on X.
$\mathrm{Pic}^{\underline{d}}(X) \cong \mathrm{Jac}(X)$ for all $\underline{d} \in \mathbb{Z}^{V(G)}$. It is not complete.

There exist compactifications over $\overline{\mathcal{M}}_g$ for all $d \in \mathbb{Z}$

$$\psi_d : \overline{\mathcal{P}}_g^d \longrightarrow \overline{\mathcal{M}}_g$$

$\overline{P}_X^d := \psi_d^{-1}(X)$ is the degree-d compactified Jacobian.
If X is smooth $\overline{P}_X^d \cong \mathrm{Pic}^d(X)$.

Compactified Jacobian in degree g for a curve X

G dual graph of X and $E(G)$ = nodes of X.

For $F \subset E(G)$ let X_F^ν be the desingularization of X at F.

Case $d = g$ (Néron compactified Jacobian).
Recursive stratification [C-Christ]:

$$\overline{P}_X^g = \bigsqcup_{O_H \in \overline{\mathcal{OP}}_G^1} \mathrm{Pic}^{\underline{d}^{O_H}}(X_{F_H}^\nu)$$

Compactified Jacobian in degree g for a curve X

G dual graph of X and $E(G) =$ nodes of X.

For $F \subset E(G)$ let X_F^ν be the desingularization of X at F.

Case $d = g$ (Néron compactified Jacobian).
Recursive stratification [C-Christ]:

$$\overline{P}_X^g = \bigsqcup_{O_H \in \overline{\mathcal{OP}}_G^1} \mathrm{Pic}^{\underline{d}^{O_H}}(X_{F_H}^\nu)$$

$\overline{\mathcal{OP}}_G^1 =$ graded poset of *rooted orientations* on (connected) spanning subgraphs of G, up to equivalence.
O_H is the class of a rooted orientation on $H = G - F_H$.

Compactified Jacobian in degree g for a curve X

G dual graph of X and $E(G) =$ nodes of X.

For $F \subset E(G)$ let X_F^ν be the desingularization of X at F.

Case $d = g$ (Néron compactified Jacobian).
Recursive stratification [C-Christ]:

$$\overline{P}_X^g = \bigsqcup_{O_H \in \overline{\mathcal{OP}}_G^1} \mathrm{Pic}^{\underline{d}^{O_H}}(X_{F_H}^\nu)$$

$\overline{\mathcal{OP}}_G^1 =$ graded poset of *rooted orientations* on (connected) spanning subgraphs of G, up to equivalence.
O_H is the class of a rooted orientation on $H = G - F_H$.

Only parts of the program are ok. [Baker-Rabinoff, Christ]

Compactified universal degree-g Jacobians

The recursive stratification over $\overline{\mathcal{M}}_g$.

$$\begin{array}{ccccl}
\overline{\mathcal{P}}^g_g & \xrightarrow{\sigma_P} & \overline{\mathcal{OP}}^1_g & & \textit{rooted orientations} \\
\Big\downarrow \psi_g & & \Big\downarrow & & \\
 & & \mathcal{H}^1_g & & \textit{connected subgraphs} \\
 & & \Big\downarrow & & \\
\overline{\mathcal{M}}_g & \xrightarrow{\sigma_M} & \mathcal{G}_g & &
\end{array}$$

[C-Christ]

Compactified Néron models and degree-g Jacobians

$$
\begin{array}{ccccl}
\overline{\mathcal{P}}_g^g & \xrightarrow{\sigma_P} & \overline{\mathcal{OP}}_g^1 & & \textit{rooted orientations} \\
 & \searrow^{\sigma_N} & \downarrow & & \\
\downarrow \psi_g & & \mathcal{H}_g^1 & & \textit{connected subgraphs} \\
 & & \downarrow & & \\
\overline{\mathcal{M}}_g & \xrightarrow{\sigma_M} & \mathcal{G}_g & &
\end{array}
$$

The recursive stratification σ_N is via Néron models of all connected partial desingularizations of X.
$N_X =$ special fiber of Néron model.

$$\overline{P}_X^g = \overline{N}_X = \bigsqcup_{H \in \mathcal{H}_G^1} N_{X^{\nu}_{F_H}}$$

Compactified Jacobian in degree $g-1$ for a curve X

G dual graph of X, and $E(G) =$ nodes of X.

For $F \subset E(G)$ let X_F^ν be the desingularization of X at F.

Case $d = g-1$. The recursive stratification

$$\overline{P}_X^{g-1} = \bigsqcup_{O_H \in \overline{\mathcal{OP}}_G^0} \mathrm{Pic}^{\underline{d}^{O_H}}(X_{F_H}^\nu)$$

Compactified Jacobian in degree $g-1$ for a curve X

G dual graph of X, and $E(G) =$ nodes of X.

For $F \subset E(G)$ let X_F^ν be the desingularization of X at F.

Case $d = g-1$. The recursive stratification

$$\overline{P}_X^{g-1} = \bigsqcup_{O_H \in \overline{\mathcal{OP}}_G^0} \text{Pic}^{\underline{d}^{O_H}}(X_{F_H}^\nu)$$

$\overline{\mathcal{OP}}_G^0$ graded poset of *totally cyclic* orientations on (bridgeless) spanning subgraphs of G, up to equivalence.
O_H is the class of a totally cyclic orientation on $H = G - F_H$.

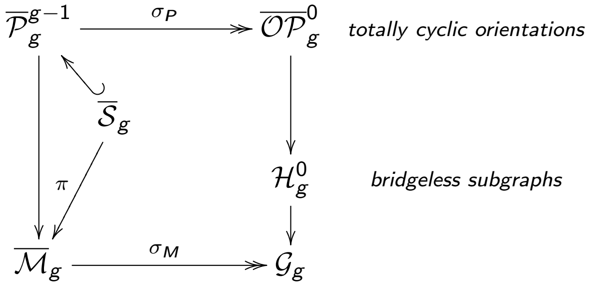
[Beauville, Alexeev, C-Viviani]

Compactified universal Jacobian in degree $g - 1$

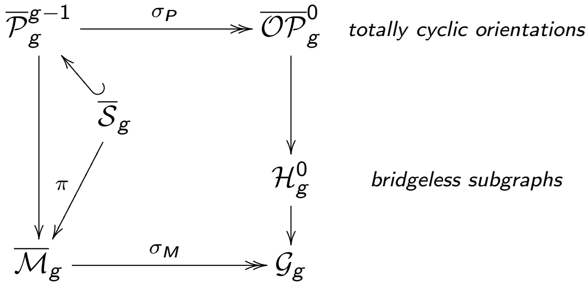
$$
\begin{array}{ccccc}
\overline{\mathcal{P}}_g^{g-1} & \xrightarrow{\sigma_P} & \overline{\mathcal{OP}}_g^{0} & & \textit{tot.cyc. orientations} \\
 & & \downarrow & & \\
\psi_{g-1} \Big\downarrow & & \mathcal{H}_g^0 & & \textit{bridgeless subgraphs} \\
 & & \downarrow & & \\
\overline{\mathcal{M}}_g & \xrightarrow{\sigma_M} & \mathcal{G}_g & &
\end{array}
$$

Only the first step of the program is known. [C-Christ]

Moduli of stable spin curves



Moduli of stable spin curves



$\overline{\mathcal{S}}_g$ moduli of *stable spin curves* [Cornalba, Jarvis, Fontanari].
$\mathcal{S}_g$ = Moduli stack of *theta characteristics*
$= \{(X, L) : X \in \overline{\mathcal{M}}_g, L \in \mathrm{Pic}(X) : L^2 \cong \omega_X\}$.

Program applies to theta characteristics/stable spin curves

$S_X = \{L \in \mathrm{Pic}(X) : L^2 \cong \omega_X\}$ = theta-characteristics on X.

$$\overline{S}_X = \pi^{-1}(X) = \bigsqcup_{G-F \in \mathcal{C}_G} S_{X_F^\nu}$$

C_G the cycle space of G (subgraphs of even degree) [C-Casagrande].

Program applies to theta characteristics/stable spin curves

$S_X = \{L \in \text{Pic}(X) : L^2 \cong \omega_X\} = \text{theta-characteristics on } X.$

$$\overline{S}_X = \pi^{-1}(X) = \bigsqcup_{G-F \in \mathcal{C}_G} S_{X_F^\nu}$$

C_G the cycle space of G (subgraphs of even degree) [C-Casagrande].

$\mathcal{SP}_g$ *stable spin graphs* of genus g.

The recursive stratification

$$\sigma_S : \overline{\mathcal{S}}_g \longrightarrow \mathcal{SP}_g; \qquad (X_F^\nu, L_F) \mapsto (G, G-F, s) \text{ dual spin graph}$$

Program applies to theta characteristics/stable spin curves

$S_X = \{L \in \text{Pic}(X) : L^2 \cong \omega_X\}$ = theta-characteristics on X.

$$\overline{S}_X = \pi^{-1}(X) = \bigsqcup_{G-F \in \mathcal{C}_G} S_{X_F^\nu}$$

C_G the cycle space of G (subgraphs of even degree) [C-Casagrande].

$\mathcal{SP}_g$ *stable spin graphs* of genus g.

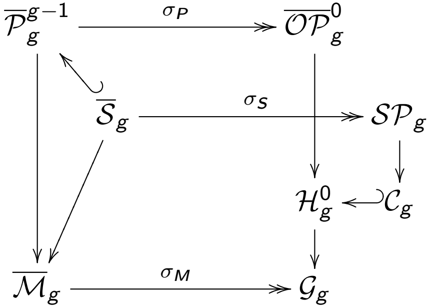
The recursive stratification

$$\sigma_S : \overline{\mathcal{S}}_g \longrightarrow \mathcal{SP}_g; \qquad (X_F^\nu, L_F) \mapsto (G, G-F, s) \text{ dual spin graph}$$

$\overline{\Sigma}(\overline{\mathcal{S}}_g) \cong \overline{S}_g^{\text{trop}}$ moduli space of *(extended) tropical spin curves* [Zharkov, C-Melo-Pacini(to appear)].

$$\text{trop} : \overline{\mathcal{S}}_g^{\text{an}} \longrightarrow \overline{\Sigma}(\overline{\mathcal{S}}_g) \xrightarrow{\cong} \overline{S}_g^{\text{trop}}$$

Spin curves and universal Jacobian in degree $g-1$



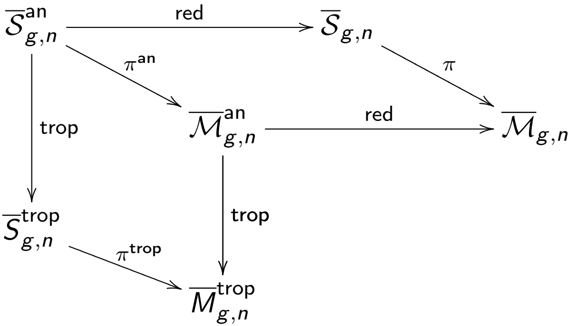
totally cyclic orientations

Spin graphs

Cyclic subgraphs

[C-Melo-Pacini(to appear)]

Program applies to n-pointed stable spin curves



Program applies to *n*-pointed stable spin curves