Recursive combinatorial aspects of compactified moduli spaces

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A moduli space, $M$, has a natural algebraic structure, not complete (i.e. degenerations must occur).

Step 1: A completion, $\overline{M}$, of $M$ has a recursive structure by topological/combinatorial type.

Step 2: There is a "cone complex", $\Sigma(M)$, associated to this structure which is also a moduli space for combinatorial objects.

Step 3: $\Sigma(M)$ is also the skeleton of the analytification, $M_{\text{an}}$, of $M$.

The retraction $\rho: M_{\text{an}} \to \Sigma(M)$ has a remarkable geometric interpretation in terms of the all moduli spaces involved.
The program in general

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The retraction $\rho : \overline{M}^{an} \longrightarrow \Sigma(\overline{M})$ has a remarkable geometric interpretation in terms of the all moduli spaces involved.
Program applies to moduli of smooth/stable curves

\[ M = \mathcal{M}_{g,n} \] is the moduli stack of smooth \( n \)-pointed curves of genus \( g \).

\[ \overline{M}_{g,n} \] is the moduli stack of stable \( n \)-pointed curves of genus \( g \).

[Deligne-Mumford, Knudsen]
$M = \mathcal{M}_{g,n}$ moduli stack of smooth $n$-pointed curves of genus $g$.

$\overline{\mathcal{M}}_{g,n}$ moduli stack of stable $n$-pointed curves of genus $g$.

[Deligne-Mumford, Knudsen]

$\mathcal{G}_{g,n}$ graded poset of stable graphs of genus $g$ with $n$ legs.

$\sigma_M : \overline{\mathcal{M}}_{g,n} \to \mathcal{G}_{g,n}; \quad X \mapsto G$ dual graph (topological type) of $X$
Program applies to moduli of smooth/stable curves

\[ M = \mathcal{M}_{g,n} \text{ moduli stack of smooth } n\text{-pointed curves of genus } g. \]

\[ \overline{\mathcal{M}}_{g,n} \text{ moduli stack of stable } n\text{-pointed curves of genus } g. \]

[Deligne-Mumford, Knudsen]

\[ G_{g,n} \text{ graded poset of stable graphs of genus } g \text{ with } n \text{ legs.} \]

\[ \sigma_M : \overline{\mathcal{M}}_{g,n} \rightarrow G_{g,n}; \quad X \mapsto G \text{ dual graph (topological type) of } X \]

The fibers of \( \sigma_M \) form a recursive stratification of \( \overline{\mathcal{M}}_{g,n} \):

\[ \overline{\mathcal{M}}_{g,n} = \bigsqcup_{G \in G_{g,n}} \mathcal{M}_G \]

\[ \mathcal{M}_G := \sigma_M^{-1}(G). \]
Curves over $\mathbb{C}$ and their dual graph: $g = 3$, $n = 0$
Stable graphs of genus 2 with no legs

The graded poset $G_{2,0}$
Program applies to moduli of smooth/stable curves

$\overline{M}_{g,n}$ moduli stack of stable $(n$-pointed, genus $g$) curves.  
$\mathcal{G}_{g,n}$ stable graphs (of genus $g$ with $n$ legs).

$$\sigma_M : \overline{M}_{g,n} \rightarrow \mathcal{G}_{g,n}; \quad X \mapsto G \text{ dual graph (topological type) of } X$$

$\Sigma(\overline{M}_{g,n})$ is the colimit of a cone complex on the graded poset $\mathcal{G}_{g,n}$. 

Program applies to moduli of smooth/stable curves

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\( \Sigma(\overline{\mathcal{M}}_{g,n}) \) is the colimit of a cone complex on the graded poset \( \mathcal{G}_{g,n} \).

\( \Sigma(\overline{\mathcal{M}}_{g,n}) \cong \overline{\mathcal{M}}^{\text{trop}}_{g,n} \) moduli space of \((\text{extended}) \ n\text{-pointed tropical curves}. \ [\text{Mikhalkin, Brannetti-Melo-Viviani, Abramovich-C-Payne}] \)
Program applies to moduli of smooth/stable curves

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$\Sigma(\overline{M}_{g,n})$ is the colimit of a cone complex on the graded poset $G_{g,n}$.

$\Sigma(\overline{M}_{g,n}) \cong \overline{M}_{g,n}^{\text{trop}}$ moduli space of (extended) $n$-pointed tropical curves. [Mikhalkin, Brannetti-Melo-Viviani, Abramovich-C-Payne]

$\overline{M}^{\text{an}}_{g,n}$ Berkovich analytification of $\overline{M}_{g,n}$, parametrizes stable curves over non-Archimedean valuation fields up to equivalence.

[Berkovich, Thuillier]
The tropicalization map

A $K$-point in $\overline{\mathcal{M}}_{g,n}^{an}$ is a stable curve $\mathcal{X}_K \to \text{Spec } K$ over the non-Archimedean valuation field $K$, up to equivalence.
The tropicalization map

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A point in $\overline{M}_{g,n}^{\text{trop}}$ is a tropical curve $(G, \ell)$ up to equivalence, where $\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$. 

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The tropicalization map \cite{Baker-Payne-Rabinoff, Tyomkin, Viviani, Abramovich-C-Payne}:

\[
trop : \overline{\mathcal{M}}_{g,n}^{\text{an}} \xrightarrow{\rho} \Sigma(\overline{\mathcal{M}}_{g,n}) \cong \overline{\mathcal{M}}_{g,n}^{\text{trop}}
\]

\[
[\mathcal{X}_K \to \text{Spec } K] \to [(G, \ell)]
\]
The tropicalization map

A $K$-point in $\overline{M}^{an}_{g,n}$ is a stable curve $\mathcal{X}_K \to \text{Spec } K$ over the non-Archimedean valuation field $K$, up to equivalence.

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The tropicalization map [Baker-Payne-Rabinoff, Tyomkin, Viviani, Abramovich-C-Payne]

$$\text{trop} : \overline{M}^{an}_{g,n} \xrightarrow{\rho} \Sigma(\overline{M}_{g,n}) \xrightarrow{\cong} \overline{M}^{\text{trop}}_{g,n}$$

$$[\mathcal{X}_K \to \text{Spec } K] \mapsto [(G, \ell)]$$

$G$ is the dual graph of the reduction, a stable curve $X$, of $\mathcal{X}_K \to \text{Spec } K$ over $\text{Spec } R$ ($R \subset K$ the valuation ring of $K$).
The tropicalization map

A $K$-point in $\overline{M}^{an}_{g,n}$ is a stable curve $X_K \to \text{Spec } K$ over the non-Archimedean valuation field $K$, up to equivalence.

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$[X_K \to \text{Spec } K] \xrightarrow{} [(G, \ell)]$

$G$ is the dual graph of the reduction, a stable curve $X$, of $X_K \to \text{Spec } K$ over $\text{Spec } R$ ($R \subset K$ the valuation ring of $K$).

$\ell : E(G) \to \mathbb{R}_{>0} \cup \infty$ measures, by the valuation of $K$, the local geometry of the total space at $X$.  

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Program applies to Hurwitz space/admissible covers

$\mathcal{H}_\bullet = \mathcal{H}_{g,h}(\pi) =$ Hurwitz space of degree-$d$ covers of a smooth curve of genus $h$ by a smooth curve of genus $g$ with exactly $b$ branch points and ramification prescribed by $\pi = (\pi_1, \ldots, \pi_b)$, with $\pi_i$ a partition of $d$. It is not complete

$\overline{\mathcal{H}}_\bullet = \overline{\mathcal{H}}_{g,h}(\pi)$ moduli stack of admissible covers.

[Harris-Mumford, Mochizuchi, Abramovich-Corti-Vistoli]
Program applies to Hurwitz space/admissible covers

\[ \mathcal{H}_\bullet = \mathcal{H}_{g,h}(\pi) = \text{Hurwitz space of degree-} d \text{ covers of a smooth curve of genus } h \text{ by a smooth curve of genus } g \text{ with exactly } b \text{ branch points and ramification prescribed by } \pi = (\pi_1, \ldots, \pi_b), \text{ with } \pi_i \text{ a partition of } d. \text{ It is not complete} \]

\[ \mathcal{H}_\bullet = \mathcal{H}_{g,h}(\pi) \text{ moduli stack of admissible covers.} \]

[Harris-Mumford, Mochizuchi, Abramovich-Corti-Vistoli]

There are two canonical morphisms, mapping an admissible cover to its target, or its source (for suitable \( m, n \in \mathbb{N} \))

\[ \overline{\mathcal{M}}_{h,m} \xleftarrow{\text{tgt}} \overline{\mathcal{H}}_{g,h}(\pi) \xrightarrow{\text{src}} \overline{\mathcal{M}}_{g,n} \]
Program applies to Hurwitz space/admissible covers

Hurwitz space \( \mathcal{H}_\bullet \subset \overline{\mathcal{H}}_\bullet = \) moduli stack of admissible covers.

\( \mathcal{A}_\bullet \) \textit{admissible covers of graphs}, a graded poset. The recursive stratification [Cavalieri-Markwig-Ranganathan]:

\[
\sigma_{\mathcal{H}} : \overline{\mathcal{H}}_\bullet \longrightarrow \mathcal{A}_\bullet ; \quad [C' \rightarrow C] \mapsto [G' \rightarrow G] \text{ dual graph cover}
\]
Program applies to Hurwitz space/admissible covers

Hurwitz space = $\mathcal{H}_\bullet \subset \overline{\mathcal{H}}_\bullet = \text{moduli stack of admissible covers.}$

$\mathcal{A}_\bullet \text{ admissible covers of graphs}$, a graded poset.

The recursive stratification \cite{Cavalieri-Markwig-Ranganathan}:

$$\sigma_{\mathcal{H}} : \overline{\mathcal{H}}_\bullet \to \mathcal{A}_\bullet; \quad [C' \to C] \mapsto [G' \to G] \quad \text{dual graph cover}$$

$\Sigma(\overline{\mathcal{H}}_\bullet)$ is the colimit of a cone complex on $\mathcal{A}_\bullet$.

$\Sigma(\overline{\mathcal{H}}_\bullet) \cong \overline{\mathcal{H}}^{\text{trop}}_\bullet$ moduli space of tropical admissible covers.

$$\text{trop} : \overline{\mathcal{H}}^\text{an}_\bullet \to \overline{\mathcal{H}}^{\text{trop}}_\bullet$$

Compatible with the program for $\overline{\mathcal{M}}_{g,n}$ via the canonical maps $\text{src}$ and $\text{tgt}$, and their tropical and analytic counterparts.
Compatibility of tropicalization maps with canonical maps

A commutative diagram

\[ \begin{array} {ccc}
\mathcal{M}_{h,m}^{\text{an}} & \xrightarrow{\text{tgt}^{\text{an}}} & \mathcal{H}^{\text{an}} \\
\downarrow \text{trop} & & \downarrow \text{trop} \\
\mathcal{M}_{h,m}^{\text{trop}} & \xrightarrow{\text{tgt}^{\text{trop}}} & \mathcal{H}^{\text{trop}} \\
\downarrow \text{trop} & & \downarrow \text{trop} \\
\mathcal{M}_{g,n}^{\text{trop}} & \xrightarrow{\text{src}^{\text{trop}}} & \mathcal{M}_{g,n}^{\text{an}}
\end{array} \]

[Cavalieri-Markwig-Ranganathan]

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Compatibly of tropicalization maps with canonical maps

A commutative diagram

$\mathcal{H}^\text{an}$

$\mathcal{M}^\text{an}_{h,m}$

$\mathcal{M}^\text{trop}_{h,m}$

$\mathcal{Trop}$

$\mathcal{M}^\text{trop}$

$\mathcal{Trop}$

$\mathcal{M}^\text{an}$

$\mathcal{M}^\text{an}_{g,n}$

$\mathcal{M}^\text{trop}_{g,n}$

$\mathcal{G}_{g,n}$

[Cavalieri-Markwig-Ranganathan]
Compatibility of tropicalization maps with canonical maps

A commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}^{an} & \xrightarrow{\text{tgt}^{an}} & \mathcal{H}^{an} \\
\downarrow \text{trop} & & \downarrow \text{trop} \\
\overline{\mathcal{M}}^{an}_{h,m} & & \overline{\mathcal{M}}^{an}_{g,n} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{H}^{trop} & \xrightarrow{\text{src}^{trop}} & \mathcal{H}^{trop} \\
\downarrow \text{tgt}^{trop} & & \downarrow \text{trop} \\
\overline{\mathcal{M}}^{trop}_{h,m} & & \overline{\mathcal{M}}^{trop}_{g,n} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sigma_M} & \mathcal{M}^{g,n} \\
\downarrow \sigma_H & & \downarrow \sigma_H \\
\overline{\mathcal{M}}^{g,n} & & \overline{\mathcal{M}}^{g,n} \\
\end{array}
\]

\[\begin{array}{ccc}
\mathcal{H}^{an} & \xrightarrow{\text{src}^{an}} & \mathcal{H}^{an} \\
\downarrow \text{trop} & & \downarrow \text{trop} \\
\overline{\mathcal{M}}^{an}_{h,m} & & \overline{\mathcal{M}}^{an}_{g,n} \\
\end{array}\]

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\overline{\mathcal{M}}^{trop}_{h,m} & & \overline{\mathcal{M}}^{trop}_{g,n} \\
\end{array}
\]

[161] [Cavalieri-Markwig-Ranganathan]
Moduli of line bundles on a stable curve

$X$ stable curve, $G$ its dual graph, 
$E(G) =$ nodes of $X$ and $V(G) =$ irreducible components of $X$. 
$\text{Pic}^d(X) = \bigsqcup_{|d| = d} \text{Pic}^d(X)$ is the moduli space of line bundles of degree $d$ on $X$.  
$\text{Pic}^d(X) \cong \text{Jac}(X)$ for all $d \in \mathbb{Z}^{V(G)}$. It is not complete.
Moduli of line bundles on a stable curve

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There exist compactifications over $\overline{M}_g$ for all $d \in \mathbb{Z}$

$$
\psi_d : \overline{P}_g^d \longrightarrow \overline{M}_g
$$

$\overline{P}_X^d := \psi_d^{-1}(X)$ is the degree-$d$ compactified Jacobian.

If $X$ is smooth $\overline{P}_X^d \cong \text{Pic}^d(X)$. 

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Compactified Jacobian in degree $g$ for a curve $X$

$G$ dual graph of $X$ and $E(G) =$ nodes of $X$.

For $F \subset E(G)$ let $X_F^\nu$ be the desingularization of $X$ at $F$.

Case $d = g$ (Néron compactified Jacobian).
Recursive stratification [C-Christ]:

$$\overline{P}_X^g = \bigsqcup_{O_H \in \overline{OP}_G^1} \text{Pic}^{d_{OH}}(X_F^\nu)$$
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$$\overline{P}^g_X = \bigsqcup_{O_H \in \overline{OP}^1_G} \text{Pic}^{d_OH}(X_F^\nu)$$

$\overline{OP}^1_G =$ graded poset of rooted orientations on (connected) spanning subgraphs of $G$, up to equivalence.

$O_H$ is the class of a rooted orientation on $H = G - F_H$. 

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Recursive stratification $[C$-Christ$]:$

$$\overline{P}^g_X = \bigsqcup_{O_H \in \overline{OP}_G^1} \text{Pic}^d_{O_H}(X^\nu_{F_H})$$

$\overline{OP}_G^1 =$ graded poset of rooted orientations on (connected) spanning subgraphs of $G$, up to equivalence.
$O_H$ is the class of a rooted orientation on $H = G - F_H$.

Only parts of the program are ok. $[Baker-Rabinoff, Christ]$
Compactified universal degree-$g$ Jacobians

The recursive stratification over $\overline{M}_g$.

\[
\begin{array}{c}
\overline{P}_g \xrightarrow{\sigma_P} \overline{OP}_g \\
\downarrow \psi_g \quad \downarrow \\
\overline{M}_g \xrightarrow{\sigma_M} G_g
\end{array}
\]

rooted orientations

connected subgraphs

[C-Christ]
Compactified Néron models and degree-$g$ Jacobians

The recursive stratification $\sigma_N$ is via Néron models of all connected partial desingularizations of $X$.

$N_X = \text{special fiber of Néron model.}$

$$\overline{P}_X^g = \overline{N}_X = \bigsqcup_{H \in \mathcal{H}_g^1} N_{X_{FH}^\nu}$$
Compactified Jacobian in degree \( g - 1 \) for a curve \( X \)

\( G \) dual graph of \( X \), and \( E(G) = \) nodes of \( X \).

For \( F \subset E(G) \) let \( X_F^\nu \) be the desingularization of \( X \) at \( F \).

Case \( d = g - 1 \). The recursive stratification

\[
\bar{P}_X^{g-1} = \bigsqcup_{O_H \in \bar{P}_G^0} \text{Pic}^{d_{O_H}}(X_F^\nu)
\]
Compactified Jacobian in degree $g - 1$ for a curve $X$

$G$ dual graph of $X$, and $E(G) =$ nodes of $X$.

For $F \subset E(G)$ let $X_F^\nu$ be the desingularization of $X$ at $F$.

Case $d = g - 1$. The recursive stratification

$$
\overline{P}_{X}^{g-1} = \bigsqcup_{O_H \in \overline{OP}_G} \text{Pic}^{dO_H}(X_F^\nu) $$

$\overline{OP}_G$ graded poset of \textit{totally cyclic} orientations on (bridgeless) spanning subgraphs of $G$, up to equivalence. $O_H$ is the class of a totally cyclic orientation on $H = G - F_H$.

[Beauville, Alexeev, C-Viviani]
Compactified universal Jacobian in degree \( g - 1 \)

\[
\begin{align*}
\overline{P}_g^{g-1} & \xrightarrow{\sigma_P} \overline{OP}^0_g & \text{tot.cyc. orientations} \\
\psi_{g-1} & & \psi_{g-1} \\
\overline{M}_g & \xrightarrow{\sigma_M} \overline{G}_g & \text{bridgeless subgraphs}
\end{align*}
\]

Only the first step of the program is known. [C-Christ]
Moduli of stable spin curves

$\overline{P}_g^{-1} \xrightarrow{\sigma_P} \overline{OP}_g$ totally cyclic orientations

$\overline{P}_g \xrightarrow{\sigma_P} \overline{OP}_g$

$\overline{S}_g \xrightarrow{\pi} \overline{M}_g$

$\overline{S}_g \xrightarrow{\pi} \overline{M}_g$

$\overline{H}_g^0 \xrightarrow{\sigma_M} G_g$ bridgeless subgraphs

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Moduli of stable spin curves

$\mathcal{S}_g$ moduli of stable spin curves [Cornalba, Jarvis, Fontanari].

$S_g =$ Moduli stack of theta characteristics

$= \{(X, L) : X \in \overline{M}_g, L \in \text{Pic}(X) : L^2 \cong \omega_X\}.$
Program applies to theta characteristics/stable spin curves

\[ S_X = \{ L \in \text{Pic}(X) : L^2 \cong \omega_X \} = \text{theta-characteristics on } X. \]

\[ \bar{S}_X = \pi^{-1}(X) = \bigsqcup_{G-F \in C_G} S_{X_F} \]

\( C_G \) the cycle space of \( G \) (subgraphs of even degree) [C-Casagrande].
Program applies to theta characteristics/stable spin curves

\[ S_X = \{ L \in \text{Pic}(X) : L^2 \cong \omega_X \} = \text{theta-characteristics on } X. \]

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C_G the cycle space of G (subgraphs of even degree) [C-Casagrande].

\( S\mathcal{P}_g \) stable spin graphs of genus \( g \).

The recursive stratification

\[ \sigma_S : \overline{S}_g \longrightarrow S\mathcal{P}_g; \quad (X_F^\nu, L_F) \mapsto (G, G - F, s) \text{ dual spin graph} \]
Program applies to theta characteristics/stable spin curves

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\[ \sigma_S : \overline{S}_g \longrightarrow S\mathcal{P}_g; \quad (X_F^\nu, L_F) \mapsto (G, G-F, s) \text{ dual spin graph} \]

\[ \overline{\Sigma}(\overline{S}_g) \cong \overline{S}_g^{\text{trop}} \text{ moduli space of (extended) tropical spin curves} \]

[Zharkov, C-Melo-Pacini(to appear)].

\[ \text{trop} : \overline{S}_g^{\text{an}} \longrightarrow \overline{\Sigma}(\overline{S}_g) \cong \overline{S}_g^{\text{trop}} \]
Spin curves and universal Jacobian in degree $g - 1$

\[ P_{g}^{g-1} \xrightarrow{\sigma_{P}} \overline{OP}_{g}^{0} \]
\[ S_{g} \xrightarrow{\sigma_{S}} \mathcal{SP}_{g} \]
\[ \mathcal{H}_{g}^{0} \leftarrow \mathcal{C}_{g} \]
\[ \overline{M}_{g} \xrightarrow{\sigma_{M}} G_{g} \]

[totally cyclic orientations]

[Spin graphs]

[Cyclic subgraphs]

[C-Melo-Pacini(to appear)]

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Program applies to $n$-pointed stable spin curves
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