Hitchin type moduli stacks in automorphic representation theory

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ICM 2018
Aug 8, 2018
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   (or: towards higher automorphic representations)
Hitchin moduli stack

- $G$: complex reductive group; $\mathfrak{g} = \text{Lie } (G)$.
- $X$: compact Riemann surface; $\omega_X$ : canonical bundle.

A $G$-Higgs bundle on $X$ is a pair $(\mathcal{E}, \varphi)$ where
  1. $\mathcal{E}$: a principal $G$-bundle on $X$;
  2. $\varphi$: a section of the vector bundle $\text{Ad}(\mathcal{E}) \otimes \omega_X$
     (When $G = \text{GL}_n$, $\mathcal{E} \leftrightarrow \mathcal{V}$ v.b. of rank $n$, $\varphi : \mathcal{V} \to \mathcal{V} \otimes \omega_X$).

- $\mathcal{M}_G$: the moduli stack of $G$-Higgs bundles on $X$. 
Hitchin moduli stack

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- $M_G$: the moduli stack of $G$-Higgs bundles on $X$. 
Key structures of $\mathcal{M}_G$

- (the stable part of) $\mathcal{M}_G$ admits a hyperKähler structure.
- $\mathcal{M}_G$ is essentially $T^*\text{Bun}_G$ (holomorphic symplectic).
- The Hitchin map
  
  $$f_G : \mathcal{M}_G \to A_G.$$  

  where $A_G \cong \prod_{i=1}^{d_i} \Gamma(X, \omega_X \otimes d_i)$, if $f_1, \cdots, f_r$ are the homogeneous free generators of the ring $\mathbb{C}[g]^G$ of degree $d_1, \cdots, d_r$.

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Group version of $\mathcal{M}_G$ (relevant to the Arthur-Selberg trace formula).

Note that

$$\mathcal{M}_G = \{ s : X \to [g/G \times \mathbb{G}_m] \mid \cdots \}.$$  

Replace $[g/G \times \mathbb{G}_m]$ by other quotient stacks? (relevant to the relative trace formula)

For example, replace $[g/G \times \mathbb{G}_m]$ by

$$(\text{End}(V) \times V^*)/(\text{GL}(V) \times \mathbb{G}_m)$$

The moduli stack of maps from $X$ are related to the Hilbert scheme of points on planar curves. ($\rightsquigarrow$ Maulik-Y., Macdonald formula....)
Variants

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\( M_G \) is essentially \( T^* \text{Bun}_G \).

Geometric Langlands: Beilinson-Drinfeld, Laumon, \ldots.

Char \( p \): Bezrukavnikov-Braverman, T.H.Chen-X.Zhu.

Global Springer theory with applications to double affine Hecke algebras (Y., Oblomkov-Y.)

Applications to classical Langlands: Ngô’s proof of the fundamental lemma (most relevant to this talk).
§2 Hitchin moduli stacks and the Arthur-Selberg trace formula
Automorphic forms

- $X/k = \mathbb{F}_q$ curve (projective, smooth, geometrically connected).
- Global function field $F = k(X)$.
- Local function field $F_x \supset \mathcal{O}_x \rightarrow k(x)$; $\wp_x$ a uniformizer.
- Ring of adèles $\mathbb{A} = \prod_{x \in |X|} F_x$.
- $G$: split reductive group over $k$.
- **Automorphic forms**: smooth functions on $G(F) \backslash G(\mathbb{A})$.
- Cuspidal automorphic representations (when $G$ is semisimple): irreducible $G(\mathbb{A})$-subrepresentations of $C^\infty_c(G(F) \backslash G(\mathbb{A}))$. 
Hecke operators

- Fix a compact open subgroup $K \subset G(\mathbb{A})$ (level).
- The space $\mathcal{A}_K = C_c(G(F) \backslash G(\mathbb{A})/K)$ is acted upon by the Hecke algebra $\mathcal{H}_K = C_c^\infty(K \backslash G(\mathbb{A})/K)$.
- Goal: understand the $\mathcal{H}_K$-module $\mathcal{A}_K$.
- The action of $f = 1_K g_K \in \mathcal{H}_K$ on $\mathcal{A}_K$ is given by the pull-push operator $q_0 \circ p_0^*$ along the correspondence

$$G(F) \backslash G(\mathbb{A})/(K \cap gKg^{-1})$$

$p_0$: natural projection; $q_0$: right multiplication by $g$. 

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The space $\mathcal{A}_K = C_c(G(F) \backslash G(\mathbb{A}) / K)$ is acted upon by the Hecke algebra $\mathcal{H}_K = C_c^K(K \backslash G(\mathbb{A}) / K)$.

Goal: understand the $\mathcal{H}_K$-module $\mathcal{A}_K$.

The action of $f = 1_{KgK} \in \mathcal{H}_K$ on $\mathcal{A}_K$ is given by the pull-push operator $q_0!p_0^*$ along the correspondence

$$G(F) \backslash G(\mathbb{A}) / (K \cap gKg^{-1})$$

$p_0$ : natural projection; $q_0$ : right multiplication by $g$. 
Arthur-Selberg trace formula

Let \( f \in \mathcal{H}_K \). Arthur-Selberg trace formula expresses the trace of \( f \) on \( \mathcal{A}_K \) in two different ways:

- The geometric expansion

\[
\text{Tr}(f, \mathcal{A}_K) = \sum_{\gamma \in G(F)/\sim} J_\gamma(f).
\]

Here, \( J_\gamma(f) \) is the orbital integral

\[
J_\gamma(f) = \text{vol}(G_\gamma(F) \backslash G_\gamma(\mathbb{A}), \mu_{G_\gamma}) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1} \gamma g) \frac{\mu_G}{\mu_{G_\gamma}}(g)
\]

\( \mu_G \) and \( \mu_{G_\gamma} \) are Haar measures on \( G(\mathbb{A}) \) and \( G_\gamma(\mathbb{A}) \).

- The spectral expansion: a sum over automorphic representations.

\(^1\)Ignore divergence issue.
Let $f \in \mathcal{H}_K$. Arthur-Selberg trace formula expresses the trace of $f$ on $\mathcal{A}_K$ in two different ways:\footnote{Ignore divergence issue.}:

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$$J_\gamma(f) = \text{vol}(G_\gamma(F) \backslash G_\gamma(\mathbb{A}), \mu_{G_\gamma}) \int_{G_\gamma(\mathbb{A}) \backslash \overline{G(\mathbb{A})}} f(g^{-1} \gamma g) \frac{\mu_G}{\mu_{G_\gamma}}(g)$$

where $\mu_G$ and $\mu_{G_\gamma}$ are Haar measures on $G(\mathbb{A})$ and $G_\gamma(\mathbb{A})$.

**The spectral expansion:** A sum over automorphic representations.
Geometric interpretation

Goal:

\[ G(F) \backslash G(\mathbb{A})/K \xleftarrow{p_0} G(F) \backslash G(\mathbb{A})/K \cap gKg^{-1} \xrightarrow{q_0} G(F) \backslash G(\mathbb{A})/K \]

Geometrize

\[ \text{Bun}_{G,K} \xleftarrow{p} \text{Hk}_{G,KgK} \xrightarrow{q} \text{Bun}_{G,K} \]
Geometric interpretation

Let $K_0 = \prod_{x \in |X|} G(O_x)$. Weil’s observation: full embedding of groupoids

$$G(F) \backslash G(\mathbb{A}) / K_0 \hookrightarrow \text{Bun}_G(k) = \{\text{principal } G\text{-bundles over } X\}.$$

When $G$ is split, this is in fact an equivalence.

Level structures: for a compact open $K = \prod_{x \in |X|} K_x \subset K_0$, $G(F) \backslash G(\mathbb{A}) / K$ is equivalent to the groupoid of $G$-bundles on $X$ with $K$-level structures.
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Double coset $K g K \subset G(\mathbb{A})$ specifies the relative position of two $G$-bundles with $K$-level structures whose generic fibers are identified.

Example: $G = \text{GL}_n$, $K = K_0$, $g = (g_x)$ where $g_x = \text{diag}(\varpi_x, 1, 1, \cdots, 1)$ for $x = x_0$ and $g_x = 1$ otherwise.

$\mathcal{V}$ and $\mathcal{V}'$: two vector bundles of rank $n$ ($\text{GL}_n$-bundles $\iff$ rank $n$ vector bundles).

A rational map $\varphi : \mathcal{V} \rightarrow \mathcal{V}'$ is in relative position $K g K \iff \varphi$ extends to an injective map of coherent sheaves $\mathcal{V} \rightarrow \mathcal{V}'$ whose cokernel is the skyscraper $k(x_0)$. 
Geometric interpretation

We introduce some moduli stacks.

- \textbf{Bun}_{G,K}: moduli stack of \(G\)-bundles on \(X\) with \(K\)-level structures.
- \textbf{Hk}_{G,KgK}: moduli stack of triples \((\mathcal{E}, \mathcal{E}', \alpha)\) where \(\mathcal{E}, \mathcal{E}'\) are \(G\)-bundles with \(K\)-level structures on \(X\), and \(\alpha: \mathcal{E} \rightarrow \mathcal{E}'\) is a \textit{rational} isomorphism between \(\mathcal{E}\) and \(\mathcal{E}'\) with relative position given by \(KgK\).
- Hecke correspondence diagram

\[
\text{Bun}_{G,K} \xleftarrow{p} \text{Hk}_{G,KgK} \xrightarrow{q} \text{Bun}_{G,K}
\]

\[
\mathcal{E} \xleftarrow{\mathcal{E}, \mathcal{E}', \alpha} \xrightarrow{} \mathcal{E}'
\]

Taking \(k\)-points, recover the previous Hecke diagram.
Define the stack $\mathcal{M}_{G,KgK}$ by the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{M}_{G,KgK} & \longrightarrow & Hk_{G,KgK} \\
\downarrow & & \downarrow (p,q) \\
\text{Bun}_{G,K} & \xrightarrow{\Delta} & \text{Bun}_{G,K} \times \text{Bun}_{G,K}
\end{array}
$$

"Trace equals integrating the kernel function over the diagonal" $\Rightarrow$

$$\text{Tr}(1_{KgK}, A_K) = \# \mathcal{M}_{G,KgK}(k).$$
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\( \mathcal{M}_{G,KgK} \) and the trace formula

- \( \mathcal{M}_{G,KgK} \) classifies \((\mathcal{E}, \varphi)\) where \( \mathcal{E} \) is a \( G \)-bundle over \( X \) with \( K \)-level structures, and \( \varphi : \mathcal{E} \to \mathcal{E} \) is a \text{rational automorphism} with relative position given by \( KgK \).
- \( \mathcal{M}_{G,KgK} \) is a group version of \( \mathcal{M}_G \).
- How to see the geometric expansion geometrically?
- It comes from an analogue of the Hitchin map

\[
h_G : \mathcal{M}_{G,KgK} \to B_{G,KgK}.
\]

sending \((\mathcal{E}, \varphi)\) to invariants of \( \varphi_\eta \); \( B_{G,KgK} \) an affine scheme over \( k \).
\[ \mathcal{M}_{G,KgK} \text{ and the trace formula} \]

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**Hitchin map**

- Grothendieck-Lefschetz trace formula implies for any $b \in \mathcal{B}_{G,KgK}(k)$

$$
\sum_{\gamma \in G(F)/\sim, \chi(\gamma) = b} J_\gamma(1_{KgK}) = \# h_G^{-1}(b) = \text{Tr}(\text{Frob}_b, (R h_! \mathbb{Q}_\ell)_b).
$$

stable orbital integrals $\leftrightarrow$ cohomology

Summing over $b \in \mathcal{B}_{G,KgK}(k)$ gives the geometric expansion of the trace.

- Such an interpretation of the orbital integrals is the starting point of B.C.Ngô’s proof of the Langlands-Shelstad fundamental lemma.
- For more, see E.Frenkel, B-C.Ngo, Geometrization of trace formulas.
§3 Hitchin moduli stacks and relative trace formulae
Let $H_1, H_2$ be subgroups of $G$.

$$
\begin{array}{c}
H_1 \xrightarrow{\varphi_1} G \xleftarrow{\varphi_2} H_2 \\
H_1(F) \backslash H_1(\mathbb{A}) / K_1 \xrightarrow{\varphi_1} G(F) \backslash G(\mathbb{A}) / K < \xleftarrow{\varphi_2} H_2(F) \backslash H_2(\mathbb{A}) / K_2
\end{array}
$$

(Here $K_i = H_i(\mathbb{A}) \cap K$.)

The relative trace of $f \in \mathcal{H}_K$ with respect to $(H_1, H_2)$:

$$
\text{RTr}^G_{H_1, H_2}(f) = \langle \varphi_1,!1, f \cdot \varphi_2,!1 \rangle_{L^2(G(F) \backslash G(\mathbb{A}) / K)}
$$

Here $\varphi_i,!1$ is the pushforward of the constant function (measure) along $\varphi_i$. 

Variant of Hitchin moduli

- $\text{RTr}^G_{H_1, H_2}(f)$ also has:
  1. geometric expansion: (relative) orbital integrals;
  2. spectral expansion: periods of automorphic forms.

Geometric interpretation

\[ \mathcal{M}_{G, KgK} \sim \mathcal{M}_G^{H_1, H_2, KgK} \]

the latter classifies $(\mathcal{E}_1, \mathcal{E}_2, \alpha)$ where

1. $\mathcal{E}_i$: an $H_i$-bundle with $K_i$-structure over $X$ for $i = 1, 2$;
2. $\alpha$: a rational isomorphism between the $G$-bundles induced from $\mathcal{E}_1$ and $\mathcal{E}_2$, with relative position given by $KgK$.

- Special case ($G = G_1 \times G_1, H_1 = H_2 = \Delta(G_1)$) recovers the Arthur-Selberg trace for $G_1$. 
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- Geometric interpretation

$$\mathcal{M}_{G,KgK} \rightarrow \mathcal{M}_{H_1,H_2,KgK}^G$$

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The Jacquet-Rallis relative trace formula

- Motivation: Gan-Gross-Prasad conjecture for unitary groups.
  (Branching law)

- Jacquet and Rallis consider two situations. Let $F$ be a global field, $E/F$ a quadratic extension ($\eta: \mathbb{A}_F^\times \to \{\pm 1\}$ corresponds to $E/F$.)

1. $G = R_{E/F} \text{GL}_n \times R_{E/F} \text{GL}_{n-1}$,
   $H_1 = \Delta(R_{E/F} \text{GL}_{n-1})$, 
   $H_2 = \text{GL}_n \times \text{GL}_{n-1}$ (all over $F$).
   The constant function on $H_2(\mathbb{A}_F)$ is replaced by the character $\eta \circ \text{det}$ on $\text{GL}_{n-1}(\mathbb{A}_F)$.

2. $G' = U_n \times U_{n-1}$,
   $H'_1 = H'_2 = \Delta(U_{n-1})$.
   Here $U_{n-1}$ is the unitary group associated with a Hermitian $E$-space $V_{n-1}$ of dimension $n-1$; $U_n$ is the one associated to $V_n = V_{n-1} \oplus E \cdot e_n$, $(e_n, e_n) = 1$.

- Compare $\text{RTr}_{H_1, H_2}^G$ and $\text{RTr}_{H'_1, H'_2}^{G'}$. 

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The Jacquet-Rallis orbital integrals

- The local orbital integral relevant to the first situation is

\[ J_{x,\gamma}^{\text{GL}}(f) := \int_{\text{GL}_{n-1}(F_x)} f(h^{-1}\gamma h)\eta_x(\det h)dh, \gamma \in S_n(F_x), \]

\[ f \in C^\infty_c(S_n(F_x)), \text{ where } S_n = \{ g \in R_{E/F}\text{GL}_n|\sigma(g) = g^{-1} \}. \]

- The local orbital integral relevant to the second situation is

\[ J_{x,\delta}^U(f) = \int_{\text{U}_{n-1}(F_x)} f(h^{-1}\delta h)dh, \quad \delta \in \text{U}_n(F_x), f \in C^\infty_c(\text{U}_n(F_x)). \]
The fundamental lemma

**Theorem (Y., 2011)**

Assume $F$ is a function field, $x$ is a place of $F$ such that $E_x/F_x$ is unramified and the Hermitian space $V_{n,x}$ has a self-dual lattices $\Lambda_{n,x}$. Then for strongly regular semisimple elements $\gamma \in S_n(F_x)$ and $\delta \in U_n(F_x)$ with the same invariants, we have

$$J_{x,\gamma}^{\text{GL}}(1_{S_n(\mathcal{O}_x)}) = \pm J_{x,\delta}^{\text{U}}(1_{U(\Lambda_{n,x})})$$

for an explicitly defined sign depending on the invariants of $\gamma$.

Proof: first reduce to the Lie algebra version, then study the geometry of Hitchin-like fibrations attached to $\mathcal{M}_{H_1,H_2}^G$ and $\mathcal{M}_{H'_1,H'_2}^{G'}$. 

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J.Gordon has extended the above theorem to mixed characteristic local fields, using model theory.

W.Zhang used the J-R relative trace formulae and the fundamental lemma proved above to prove the Gan-Gross-Prasad conjecture for unitary groups (under some local conditions).
§4 Hitchin moduli stacks and Shtukas
Drinfeld introduced the notion of Shtukas, generalizing his elliptic modules. He used Shtukas to prove the global Langlands correspondence for $GL_2$ over function fields.

Varshavsky introduced $G$-Shtukas for a reductive group $G$.

L.Lafforgue used Drinfeld’s Shtukas to prove the global Langlands correspondence for $GL_n$ over function fields.

V.Lafforgue used $G$-Shtukas to prove the “automorphic to Galois” direction of the global Langlands correspondence for reductive groups $G$ over function fields.
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Definition of moduli of Shtukas

The definition is similar to that of $\mathcal{M}_{G,KgK}$: just replace the diagonal map by the graph of Frobenius.

$$\begin{align*}
\text{Sht}^\mu_{G,K} & \rightarrow Hk^\mu_{G,K} \\
\downarrow & \downarrow \\
\text{Bun}_{G,K} & \rightarrow \text{Bun}_{G,K} \times \text{Bun}_{G,K}
\end{align*}$$

- $r$: the number of legs.
- $\mu = (\mu_1, \cdots, \mu_r)$, $\mu_i$ dominant coweights,
- $Hk^\mu_{G,K}$ classifies chains of modifications at legs $x_i$

$$\mathcal{E}_0 - \frac{f_1}{x_1} > \mathcal{E}_1 - \frac{f_2}{x_2} > \cdots - \frac{f_r}{x_r} > \mathcal{E}_r$$

where the relative position of $f_i$ is bounded by $\mu_i$, $1 \leq i \leq r$. 
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\text{Sht}^\mu_{G,K} & \longrightarrow \text{Hk}^\mu_{G,K} \\
\downarrow & \quad \downarrow (p_0,p_r) \\
\text{Bun}_{G,K} & \longrightarrow \text{Bun}_{G,K} \times \text{Bun}_{G,K}
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Basic structures and properties

- Key structure: recording the legs gives a map

$$\text{Sht}_{G,K}^{\mu} \rightarrow X^r$$

- When all $r = 0$ (so $\mu = \emptyset$), $\text{Sht}_{G,K}^{\emptyset} = \text{Bun}_{G,K}(k)$.

- When all $\mu_i$ are minuscule, $\text{Sht}_{G,K}^{\mu}$ is smooth. (compare: Shimura varieties)

- $\text{Sht}_{G,K}^{\mu}$ is a Deligne-Mumford stack over $k = \mathbb{F}_q$ of dimension $\sum_{i=1}^r (\langle 2\rho, \mu_i \rangle + 1)$. It is locally of finite type but usually not of finite type (main difficulty compared to Shimura varieties).

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Basic structures and properties

- Key structure: recording the legs gives a map

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Higher automorphic forms are cohomology classes of $\text{Sht}^\mu_{G,K}$ for general $\mu$.

- Similarity: both have Hecke actions.
- Why higher? Automorphic forms appear in $H^*(\text{Sht}^\mu_{G,K})$ with interesting multiplicity spaces which carry information about the Galois side.
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Periods of automorphic representations

- Period of an automorphic representation $\pi$ of $G(\mathbb{A})$ along a subgroup $H$ is the $H(\mathbb{A})$-invariant linear functional

$$\mathcal{P}^G_{H,\pi} : \pi \rightarrow \mathbb{C}$$

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Periods are related to special values of $L$-functions.

- Example: $G = \text{GL}_2$, $H = \text{diagonal torus}$ $\rightsquigarrow$ the standard $L$-function of $\pi$.
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Higher periods

- Let $H$ be a reductive subgroup of $G$. If the choices of $\lambda$ and $\mu$ satisfy certain root-theoretic conditions, there is a natural map

$$\theta : \text{Sht}_{H}^{\lambda} \rightarrow \text{Sht}_{G}^{\mu}.$$ 

- Under further conditions, $\theta^*$ induces a map on intersection cohomology and defines a linear map

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Definition

Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A})$. Restricting $P_{H,\lambda}^{G,\mu}$ to the $\pi$-part

$$P_{H,\lambda,\pi}^{G,\mu} : \text{IH}_{c}^{2d_{H}(\lambda)}(\text{Sht}_{G}^{\mu} \otimes \overline{k})[\pi] \rightarrow \overline{Q}_{\ell}$$

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Zhiwei Yun (MIT)  Hitchin moduli and automorphic reps  ICM 2018  31 / 37
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Higher periods are related to higher derivatives of $L$-functions.

Let $\pi$ be a (sufficiently general) cuspidal automorphic representation of $G(\mathbb{A})$. The $\pi$-part of $\text{IH}^2_{2dH}(\lambda)(\text{Sht}^\mu_G \otimes \overline{k})$ is expected to be

$$\pi \otimes (\otimes_{i=1}^r H^1(X \otimes \overline{k}, j!\star \rho^\mu_{\pi_i})).$$

1. $\pi \mapsto \rho_\pi$ is the global Langlands correspondence ($\rho_\pi$ is a $\hat{G}$-local system on Spec $F$);
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Higher Gross-Zagier formula – setup

- $G = \text{PGL}_2$, no level structure $K = K_0 = \prod_x G(\mathcal{O}_x)$.
- $r \geq 0$ an even integer.
- $\mu = (\mu_1, \cdots, \mu_r)$, where each $\mu_i$ is the minuscule coweight for $G$. We have the moduli stack $\text{Sht}^r_G := \text{Sht}^\mu_{G,K_0}$.
- $X'/X$: an unramified double cover.
- $\text{Sht}^\lambda_T := \text{Sht}^\lambda_{\text{GL}_1,X'}/\text{Pic}_X(k)$, the moduli of rank one Shtukas on $X'$ with modification type $\lambda = (\lambda_1, \cdots, \lambda_r)$ ($\lambda_i = \pm 1$, $\sum_i \lambda_i = 0$), modulo twisting by line bundles from $X$. 
Higher Gross-Zagier formula – setup

- Natural map

\[ \theta : \text{Sht}_T^\lambda \rightarrow \text{Sht}_{G}^{lr} := \text{Sht}_{G}^r \times X^r \times X^{lr}. \]

\[ \dim : r \rightarrow 2r \]

The image of \( \theta \) gives the Heegner-Drinfeld cycle

\[ Z^\lambda \in H^2_{c} (\text{Sht}_{G}^{lr} \otimes \overline{k}, \mathbb{Q}_{\ell}(r)). \]

- \( \pi \): everywhere unramified cuspidal automorphic representation \( \pi \) of \( G(\mathbb{A}) \),

- \( Z^\lambda_{\pi} \): the projection of \( Z^\lambda_{\pi} \) onto the \( \pi \)-isotypic part \( H^2_{c} (\text{Sht}_{G}^{lr} \otimes \overline{k}, \mathbb{Q}_{\ell}(r))[\pi] \).

making sense of this requires serious work, because the cohomology is \( \infty \)-dimensional.
Higher Gross-Zagier formula – unramified version

**Theorem (Y.-Zhang, 2015)**

We have

\[
\langle Z^\lambda_{\pi}, Z^\lambda_{\pi} \rangle_{\text{Sh}_{G}^{r}} = \frac{q^{2-2g}}{2(\log q)^{r}} \frac{L^{(r)}(\pi_{F'}, 1/2)}{L(\pi, \text{Ad}, 1)}
\]

where

- \(\langle Z^\lambda_{\pi}, Z^\lambda_{\pi} \rangle_{\text{Sh}_{G}^{r}}\) is the self-intersection number of the cycle class \(Z^\lambda_{\pi}\).
- \(\pi_{F'}\) is the base change of \(\pi\) to \(F' = k(X')\).
- \(L(\pi_{F'}, s) = q^{4(g-1)(s-1/2)}L(\pi_{F'}, s)\) is the normalized \(L\)-function of \(\pi_{F'}\) such that \(L(\pi_{F'}, s) = L(\pi_{F'}, 1 - s)\).
- \(L^{(r)}(\pi_{F'}, 1/2)\) is the \(r\)-th derivative of \(L(\pi_{F'}, s)\) at \(s = 1/2\).
Remarks on the theorem

- $r = 0$: unramified case of the Waldspurger formula relating toric periods of $\pi$ with central $L$-values.
- There is a version of the above theorem (Y.-Zhang, 2017) which allows $\pi$ to have square-free levels, and $X'/X$ is allowed to be ramified.
  In this generalization, $(-1)^r$ is the same as the sign of the functional equation for $L(\pi_{F'}, s)$.
- In particular, for $r = 1$ we get a function field analogue of the classical Gross-Zagier formula.
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- In particular, for \( r = 1 \) we get a function field analogue of the classical Gross-Zagier formula.
Comments on the proof of higher GZ

- Proof does not involve computing either side.
- Key part of the proof: comparing two relative traces

\[
I(f) = \langle Z^\lambda, f \cdot Z^\lambda \rangle_{\text{Sht}_G^r}
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\[
J(f, s) = \langle 1_{[\text{Pic}_X(k)]}, \eta \cdot [\text{Pic}_X(k)]^s \rangle_{G(F) \setminus G(\mathbb{A})/K_0}.
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I(f) \sim J^{(r)}(f, 0).
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- Hitchin type moduli spaces play a key role in the proof. Their appearance is not due to the similarity between the moduli of Shtukas and the Hitchin stack, but “orthogonal” to this similarity.

- Similar geometric ideas can be used to prove the function field version of W. Zhang’s Arithmetic Fundamental Lemma (in progress), which related derivatives of (Jacquet-Rallis) orbital integrals to intersection numbers in the moduli of unitary local Shtukas.
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