Algebraic surfaces with minimal betti numbers

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Outline

Classify algebraic varieties up to connected moduli

- Work over the field $\mathbb{C}$ of complex numbers.

Algebraic varieties of dimension 1 are called algebraic curves.

Smooth algebraic curves (Riemann surfaces) are classified by the " mighty" genus

$g(C) :=$ (the number of "holes" of C) $= \dim_{\mathbb{C}} H^0(C, \Omega^1_C) = \frac{1}{2} \dim_{\mathbb{Q}} H_1(C, \mathbb{Q})$.

$$g(C) = 0 \Longleftrightarrow C \cong \mathbf{P}^1 \cong (Riemann\ \ sphere) = \mathbb{C} \cup \{\infty\}.$$

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$$g(C) = 0 \iff C \cong \mathbf{P}^1 \cong (\textit{Riemann sphere}) = \mathbb{C} \cup \{\infty\}.$$

In dimension 2, have several invariants

$$p_g(X) := \dim H^2(X, \mathcal{O}_X) = \dim H^0(X, \Omega^2_X)$$

$$q(X) := \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega^1_X)$$

$$b_i(X) := \dim H^i(X, \mathbb{Q})$$

Smooth Algebraic Surfaces with $p_g = q = 0$

Long history : Castelnuovo's rationality criterion, Severi conjecture, ...

By Enriques-Kodaira classification of algebraic surfaces (1950's), such surfaces are:

- $\mathbf{P}^2$, rational ruled surfaces
- Enriques surfaces
- properly elliptic surfaces with $p_g = q = 0$
- surfaces of general type with $p_g = 0$; these have $K^2 = 1, 2, \ldots, 9$.
- blowups of the above surfaces

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Smooth algebraic surfaces with minimal invariants, that is,

$$p_g = q = 0, b_1 = b_3 = 0, b_0 = b_2 = b_4 = 1$$

- $\mathbf{P}^2$
- fake projective planes (= surfaces of general type with $p_g = 0$, $K^2 = 9$)

Remark. Exotic $\mathbf{P}^2$ does not exist in complex geometry.

$\mathbb{Q}$-homology $\mathbf{P}^2$

Definition

A normal projective surface S is called a $\mathbb{Q}$-homology $\mathbf{P}^2$ if $b_i(S) = b_i(\mathbf{P}^2)$ for all i, i.e. $b_1 = b_3 = 0, b_0 = b_2 = b_4 = 1$.

- If S is smooth, then $S = \mathbf{P}^2$ or a fake projective plane.
- If S has A_1-singularities only, then $S = \mathbf{P}^2(1, 1, 2)$.
- If S has A_2-singularities only, then S has $3A_2$ or $4A_2$ and $S = \mathbf{P}^2/G$ or FPP$/G$, where $G \cong \mathbb{Z}/3$ or $(\mathbb{Z}/3)^2$.
- If S has A_1 or A_2-singularities only, $S = \mathbf{P}^2(1, 2, 3)$ or one of the above.

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- If S has A_1 or A_2-singularities only, $S = \mathbf{P}^2(1, 2, 3)$ or one of the above.

In this talk, S has at worst quotient singularities.
Then S is a $\mathbb{Q}$-homology $\mathbf{P}^2$ if $b_2(S) = 1$.

For a minimal resolution $S' \to S$, $p_g(S') = q(S') = 0$.

Trichotomy: $K_S =$ ample, $-$ample, num. trivial

Let S be a $\mathbb{Q}$-hom $\mathbf{P}^2$ with quotient singularities.

- $-K_S$ is ample
  - log del Pezzo surfaces of Picard number 1, e.g. $\mathbf{P}^2/G$, $\mathbf{P}^2(a,b,c)$, ...
  - $\kappa(S') = -\infty$.
- K_S is numerically trivial.
  - log Enriques surfaces of Picard number 1.
  - $\kappa(S') = -\infty, 0$.
- K_S is ample.
  - e.g. quotients of fake projective planes, suitable contraction of a suitable blowup of some Enriques surface, ...
  - $\kappa(S') = -\infty, 0, 1, 2$.

Problem

Classify all $\mathbb{Q}$-homology $\mathbf{P}^2$'s with quotient singularities.

The Maximum Number of Quotient Singularities

Question

How many singular points on S, a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities?

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- $|Sing(S)| \leq 5$ by the orbifold Bogomolov-Miyaoka-Yau inequality (Sakai, Miyaoka, Megyesi for K nef, Keel-McKernan for $-K$ nef)

$$\frac{1}{3}K_S^2 \leq e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \left(1 - \frac{1}{|\pi_1(L_p)|}\right).$$

- Many examples with $|Sing(S)| \leq 4$ (cf. Brenton, 1977)
- If $-K_S$ is ample, $|Sing(S)| \leq 4$ (Belousov, 2008).

Theorem (D.Hwang-Keum, 2011)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities. Then $|Sing(S)| \leq 4$ except the following case:
S has 5 singular points of type $3A_1 + 2A_3$, and its minimal resolution S' is an Enriques surface.

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Corollary

Every $\mathbb{Z}$-homology $\mathbf{P}^2$ with quotient singularities has at most 4 singular points.

Remark

(1) Every $\mathbb{Z}$-cohomology $\mathbf{P}^2$ with quotient singularities has at most 1 singular point. If it has, then the singularity is of type E_8 [Bindschadler-Brenton, 1984].
(2) $\mathbb{Q}$-homology $\mathbf{P}^2$ with rational singularities may have arbitrarily many singularities, no bound.

Smooth $\mathbf{S}^1$-action on $\mathbf{S}^m$

$$\mathbf{S}^1 \subset \mathit{Diff}(\mathbf{S}^m).$$

The identity element $1 \in \mathbf{S}^1$ acts identically on $\mathbf{S}^m$.

Each diffeomorphism $g \in \mathbf{S}^1$ is homotopic to the identity map $1_{\mathbf{S}^m}$.
By Lefschetz Fixed Point Formula,

$$e(Fix(g)) = e(Fix(1)) = e(\mathbf{S}^m).$$

If m is even, then $e(\mathbf{S}^m) = 2$ and such an action has a fixed point, so the foliation by circles degenerates.

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Assume $m = 2n - 1$ odd.

Definition

A $\mathcal{C}^\infty$-action of $\mathbf{S}^1$ on $\mathbf{S}^{2n-1}$

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

is called a pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$ if it is free except for finitely many orbits (whose isotropy groups $\mathbb{Z}/a_1, \ldots, \mathbb{Z}/a_k$ have pairwise prime orders).

Pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$

Example

Linear actions $\mathbf{S}^{2n-1} = \{(z_1, z_2, ..., z_n) : |z_1|^2 + |z_2|^2 + ... + |z_n|^2 = 1\} \subset \mathbb{C}^n$, $\mathbf{S}^1 = \{\lambda : |\lambda| = 1\} \subset \mathbb{C}$.

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

$$(\lambda, (z_1, z_2, ..., z_n)) \to (\lambda^{a_1} z_1, \lambda^{a_2} z_2, ..., \lambda^{a_n} z_n),$$

$a_1, ..., a_n$ pairwise prime.

- In this linear case
$$\mathbf{S}^{2n-1}/\mathbf{S}^1 \cong \mathbb{CP}^{n-1}(a_1, a_2, ..., a_n).$$
- The orbit of the i-th coordinate point $e_i \in \mathbf{S}^{2n-1}$ is an exceptional orbit iff $a_i \geq 2$.
- The orbit of a non-coordinate point of $\mathbf{S}^{2n-1}$ is NOT exceptional.
- This action has at most n exceptional orbits.
- The quotient map $\mathbf{S}^{2n-1} \to \mathbb{CP}^{n-1}(a_1, a_2, ..., a_n)$ is a Seifert fibration.

Pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$

- For $n = 2$ Seifert (1932) showed that each pseudo-free $\mathbf{S}^1$-action on $\mathbf{S}^3$ is linear and hence has at most 2 exceptional orbits.
- For $n = 4$ Montgomery-Yang (1971) showed that given arbitrary collection of pairwise prime positive integers $a_1, \dots, a_k$, there is a pseudofree $\mathbf{S}^1$-action on a homotopy $\mathbf{S}^7$ whose exceptional orbits have exactly those orders.
- Petrie (1974) generalised the above M-Y for all $n \geq 5$.

Conjecture (Montgomery-Yang problem, Fintushel-Stern 1987)

A pseudo-free $\mathbf{S}^1$-action on $\mathbf{S}^5$ has at most 3 exceptional orbits.

- This problem is wide open. F-S withdrew their paper [$O(2)$-actions on the 5-sphere, Invent. Math. 1987].

- Pseudo-free $\mathbf{S}^1$-actions on a manifold Σ have been studied in terms of the orbit space $\Sigma/\mathbf{S}^1$.
- The orbit space $X = \mathbf{S}^5/\mathbf{S}^1$ of such an action is a 4-manifold with isolated singularities whose neighborhoods are cones over lens spaces $S^3/\mathbb{Z}_{a_i}$ corresponding to the exceptional orbits of the $\mathbf{S}^1$-action.

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- Easy to check that X is simply connected and $H_2(X, \mathbb{Z})$ has rank 1 and intersection matrix $(1/a_1 a_2 \cdots a_k)$.
- An exceptional orbit with isotropy type $\mathbb{Z}/a$ has an equivariant tubular neighborhood which may be identified with $\mathbb{C} \times \mathbb{C} \times \mathbf{S}^1$ with a $\mathbf{S}^1$-action

$$\lambda \cdot (z, w, u) = (\lambda^r z, \lambda^s w, \lambda^a u)$$

where r and s are relatively prime to a.

The following 1-1 correspondence was known to Montgomery-Yang, Fintushel-Stern, and revisited by Kollár(2005).

Theorem

There is a one-to-one correspondence between:

1. *Pseudo-free* $\mathbf{S}^1$*-actions on* $\mathbb{Q}$*-homology 5-spheres* Σ *with* $H_1(\Sigma, \mathbb{Z}) = 0$.
2. *Compact differentiable* 4*-manifolds* M *with boundary such that*
    1. $\partial M = \bigcup_i L_i$ *is a disjoint union of lens spaces* $L_i = S^3/\mathbb{Z}_{a_i}$,
    2. *the* a_i*'s are pairwise prime,*
    3. $H_1(M, \mathbb{Z}) = 0$,
    4. $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$.

Furthermore, Σ *is diffeomorphic to* $\mathbf{S}^5$ *iff* $\pi_1(M) = 1$.

Algebraic Montgomery-Yang Problem

This is the M-Y Problem when $\mathbf{S}^5/\mathbf{S}^1$ attains a structure of a normal projective surface.

Conjecture (J. Kollár)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with at worst quotient singularities. If $\pi_1(S^0) = \{1\}$, then S has at most 3 singular points.

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What if the condition $\pi_1(S^0) = \{1\}$ is replaced by the weaker condition $H_1(S^0, \mathbb{Z}) = 0$?

There are infinitely many examples S with $H_1(S^0, \mathbb{Z}) = 0$, $\pi_1(S^0) \neq \{1\}$, $|Sing(S)| = 4$.

These examples obtained from the classification of surface quotient singularities [E. Brieskorn, Invent. Math. 1968].

Example (coming from Brieskorn's classification)

$I_m \subset GL(2,\mathbb{C})$ the $2m$-ary icosahedral group $I_m = \mathbb{Z}_{2m}.\mathcal{A}_5$.

$$1 \to \mathbb{Z}_{2m} \to I_m \to \mathcal{A}_5 \subset PSL(2,\mathbb{C})$$

I_m acts on $\mathbb{C}^2$. This action extends naturally to $\mathbf{P}^2$. Then

$$S := \mathbf{P}^2/I_m$$

is a $\mathbb{Z}$-homology $\mathbf{P}^2$ with $-K_S$ ample,

- S has 4 quotient singularities:
  one non-cyclic singularity of type I_m (the image of $O \in \mathbb{C}^2$), and
  3 cyclic singularities of order $2,3,5$ (on the image of the line at infinity),
- $\pi_1(S^0) = \mathcal{A}_5$, hence $H_1(S^0,\mathbb{Z}) = 0$.

Call these Brieskorn quotients.

Progress on Algebraic Montgomery-Yang Problem

Theorem (D.Hwang-Keum, 2011)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities, not all cyclic, such that $\pi_1(S^0) = \{1\}$. Then $|Sing(S)| \leq 3$.

More precisely

Theorem

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with 4 or more quotient singularities, not all cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$. Then S is isomorphic to a Brieskorn quotient.

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More Progress on Algebraic Montgomery-Yang Problem:

Theorem (D.Hwang-Keum, 2013, 2014)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with cyclic singularities such that $H_1(S^0, \mathbb{Z}) = 0$. If either S is not rational or $-K_S$ is ample, then $|Sing(S)| \leq 3$.

The Remaining Case of Algebraic M-Y Problem:

S is a $\mathbb{Q}$-homology $\mathbf{P}^2$ satisfying
(1) S has cyclic singularities only,
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Yes. Examples given by

- Keel and Mckernan (Mem. AMS 1999),
- Kollár (Pure Appl. Math. Q. 2008) — an infinite series of examples with $|Sing(S)| = 2$.
- D. Hwang and Keum (Proc. AMS 2012) — infinite series of examples with $|Sing(S)| = 1, 3$.

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Problem

Are there any $\mathbb{Q}$-homology $\mathbf{P}^2$ which is a rational surface S with K_S ample and with $|Sing(S)| = 4$?

Symplectic Montgomery-Yang Problem

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Question

Bogomolov inequality holds for symplectic compact 4-manifolds?

$$c_1^2 \leq 4c_2$$

In positive characteristic

In characteristic $p > 0$, have a similar notion:

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In characteristic $p > 0$, Gorenstein $\mathbb{Q}_l$-cohomology $\mathbf{P}^2$ with $K \equiv 0$ have been classified (M. Schütt, 2016, 2017).

Remark

(1) Over $\mathbb{C}$, Gorenstein $\mathbb{Q}$-homology $\mathbf{P}^2$ with $K \equiv 0$ have been classified (Hwang-Keum-Ohashi, 2015; M. Schütt, 2015).
(2) These surfaces, in any characteristic, come from Enriques surfaces, i.e., are obtained by contracting 9 smooth rational curves on an Enriques surface.

Fake Projective Planes

A compact complex surface with the same Betti numbers as $\mathbf{P}^2$ is called a fake projective plane if it is not biholomorphic to $\mathbf{P}^2$.

A FPP has ample canonical divisor K, so it is a smooth proper (geometrically connected) surface of general type with $p_g = 0$ and $K^2 = 9$ (this definition extends to arbitrary characteristic.)

The existence of a FPP was first proved by Mumford (1979) based on the theory of 2-adic uniformization, and later two more examples by Ishida-Kato (1998) in this abstract method.

Keum (2006) gave a construction of a FPP with an order 7 automorphism, which is birational to an order 7 cyclic cover of a Dolgachev surface.

Keum FPP and Mumford FPP belong to the same class, in the sense that their fundamental groups are both contained in the same maximal arithmetic subgroup of the isometry group of the complex 2-ball.

FPP's have Chern numbers $c_1^2 = 3c_2 = 9$ and are complex 2-ball quotients by Aubin (1976) and Yau (1977). Such ball quotients are strongly rigid by Mostow's rigidity theorem (1973), that is, determined by fundamental group up to holomorphic or anti-holomorphic isomorphism.

FPP's come in complex conjugate pairs by Kharlamov-Kulikov (2002) and have been classified as quotients of the two-dimensional complex ball by explicitly written co-compact torsion-free arithmetic subgroups of $PU(2,1)$ by Prasad-Yeung (2007, 2010) and Cartwright-Steger (2010). The arithmeticity of their fundamental groups was proved by Klingler (2003).

There are exactly 100 fake projective planes total, corresponding to 50 distinct fundamental groups.

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Interesting problems on fake projective planes:

- Exceptional collections in $D^b(coh(X))$
- Bicanonical map
- Explicit equations
- Bloch conjecture on zero cycles
- Modular forms

Exceptional collections on Fake Projective Planes

Conjecture (Galkin-Katzarkov-Mellit-Shinder)

On every fake projective plane X there exists an exceptional sequence of length 3. In other words, there is an ample line bundles L on X such that

$$\mathcal{O}_X, \; -L, \; -2L$$

form an exceptional collection on X.

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The conjecture has been confirmed for several FPP's [Keum, 2013, 2017], [GKMS, 2015], [Fakhruddin, 2015].

Remark

For some FPP's, such L is not unique.

Bicanonical map of Fake Projective Planes

For a fake projective plane X, $p_g(X) = 0$ hence no canonical map.

Applying Reider's theorem (1988) on adjoint linear systems to X, $|2K_X + L|$ is very ample for any ample line bundle L on X.

E.g. the tri-canonical map

$$|3K_X| : X \to \mathbf{P}^{27}$$

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By Reider again, the bicanonical system $|2K_X|$ is base point free and

$$|2K_X| : X \to \mathbf{P}^9$$

is an embedding away from finite number of points.

Theorem (Catanese-Keum 2018)

The bi-canonical map is the normalization map of the image.
For 10 pairs of fake projective planes, the bi-canonical map is an embedding.

Explicit equations of a Fake Projective Plane

It has long been of great interest since Mumford to find equations of a projective model of a fake projective plane.

With Lev Borisov we find equations of a projective model (the bicanonical image) of a conjugate pair of fake projective planes by studying the geometry of the quotient of such surface by an order seven automorphism [Keum, 2008].

The equations are given explicitly as 84 cubics in 10 variables with coefficients in the field $\mathbb{Q}[\sqrt{-7}]$.

The complex conjugate equations define the bicanonical image of the complex conjugate of the surface.

This pair has the most geometric symmetries among the 50 pairs, in the sense that it has the large automorphism group $G_{21} = \mathbb{Z}_7 : \mathbb{Z}_3$ and the $\mathbb{Z}_7$-quotient has a smooth model of a $(2, 4)$-elliptic surface which is not simply connected. The universal double cover of this elliptic surface has the Hodge numbers of a K3 surface, but Kodaira dimension 1.

Equations of the fake projective plane 1-84

$$
\begin{aligned}
eq_1 &= U_1U_2U_3 + (1 - \mathrm{i}\sqrt{7})(U_3^2U_4 + U_1^2U_5 + U_2^2U_6) + (10 - 2\mathrm{i}\sqrt{7})U_4U_5U_6 \\
eq_2 &= (-3 + \mathrm{i}\sqrt{7})U_0^3 + (7 + \mathrm{i}\sqrt{7})(-2U_1U_2U_3 + U_7U_8U_9 - 8U_4U_5U_6) \\
&\quad + 8U_0(U_1U_4 + U_2U_5 + U_3U_6) + (6 + 2\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) \\
eq_3 &= (11 - \mathrm{i}\sqrt{7})U_0^3 + 128U_4U_5U_6 - (18 + 10\mathrm{i}\sqrt{7})U_7U_8U_9 \\
&\quad + 64(U_2U_4^2 + U_3U_5^2 + U_1U_6^2) + (-14 - 6\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) \\
&\quad + 8(1 + \mathrm{i}\sqrt{7})(U_1^2U_8 + U_2^2U_9 + U_3^2U_7 - 2U_1U_2U_3) \\
eq_4 &= -(1 + \mathrm{i}\sqrt{7})U_0U_3(4U_6 + U_9) + 8(U_1U_2U_3 + U_1U_6U_9 + U_5U_7U_9) \\
&\quad + 16(U_5U_6U_7 - U_1^2U_5 - U_3U_5^2) \\
eq_5 &= g_3(eq_4) \\
eq_6 &= g_3^2(eq_4) \\
&\ \ \vdots
\end{aligned}
$$